At this stage we can evaluate $\oint_C f(z)\,dz$ by evaluating $\oint_{C_i} dz\, f(z)$ around C_i any way that we want.

In fact we can develop a simple formula for evaluating the residues, as follows: Suppose there is a singularity at z_0 then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n} \tag{3}$$

$$a_n = \frac{1}{2\pi i} \oint_{C_0} dz' \frac{f(z')}{(z'-z_0)^{n+1}} \quad ; \quad b_n = \frac{1}{2\pi i} \oint_{C_0} dz' \frac{f(z')}{(z'-z_0)^{-n+1}} \tag{4}$$

$$\therefore \oint_{C_0} f(z)\,dz \equiv 2\pi i R_0 = \sum_{n=0}^{\infty} a_n \underbrace{\oint_{C_0} dz\,(z-z_0)^n}_{=0 \text{ by Cauchy theorem}} + \sum_{n=1}^{\infty} b_n \oint_{C_0} dz \frac{1}{(z-z_0)^n} \tag{5}$$

Let C_0 = circle of radius r_0 around z_0. Then

$$(z-z_0)^n = r_0^n e^{in\theta} \qquad dz = i r_0 e^{i\theta} d\theta \tag{6}$$

$$\oint_{C_0} \cdots = \int_0^{2\pi} (r_0^{-n} e^{-in\theta})(i r_0) e^{i\theta} d\theta = i r_0^{-n+1} \int_0^{2\pi} d\theta\, e^{-i(n-1)\theta} \tag{7}$$

<u>For $n>1$</u> : $$\int_0^{2\pi} \cdots = \frac{1}{-i(n-1)} e^{-i(n-1)\theta} \Big|_0^{2\pi} = 0 \tag{8}$$

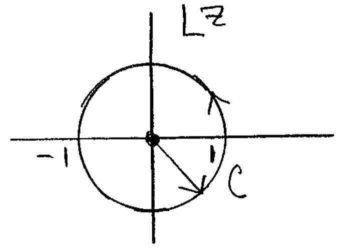
<u>For $n=1$</u> : $$\int_0^{2\pi} \cdots = i (r_0)^{-1+1} \int_0^{2\pi} d\theta = 2\pi i \tag{9}$$

$$\boxed{\begin{aligned} &\text{Hence } (5)\,\&\,(9) \Rightarrow \oint_{C_0} dz\, f(z) = 2\pi i\, b_1 \equiv 2\pi i\, R(z=z_0) \\ &\qquad \Rightarrow R(z=z_0) = b_1 = \text{coefficient of } \frac{1}{z-z_0} \end{aligned}} \tag{10}$$

Applications of Residue Theory:

[1] $I = \oint_C dz\, e^{1/z}$; C= unit circle about origin

IMPORTANT! DRAW CONTOUR WITH SINGULARITIES

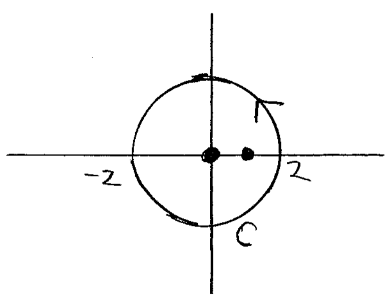


this is $\equiv b_1 =$ coeff of $\frac{1}{z-0}$

Solution: $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} = 1 + \frac{1}{(z-0)} + \frac{1}{2!(z-0)^2} + \cdots$ (11)

Hence by inspection $\boxed{\oint_C dz\, f(z) \equiv 2\pi i (b_1) = 2\pi i \cdot 1 = 2\pi i}$ (12)

[2] $I = \oint_C dz \frac{(5z-2)}{z(z-1)}$; C = circle $|z| = 2$ (13)



This contour surrounds 2 poles: at $z=0$, $z=1$

$$\oint_C dz f(z) = 2\pi i \left[\text{Residues } \underline{\text{inside}} \text{ contour}\right]$$

$$= 2\pi i \left[b_1(z=0) + b_1(z=1)\right] \quad (14)$$

At this stage we can carry out a Laurent expansion about $z=0$, and then about $z=1$. However, this can here be done <u>by inspection</u>

At $z=0$: Write $f(z)$ as $f(z) = \frac{5z-2}{z(z-1)} = \frac{(5z-2)/(z-1)}{z}$ (15)

Near $z=0$ the function $f(z)$ behaves as $f(z) \xrightarrow{z=0} \frac{(-2)/(-1)}{z} = \frac{2}{z}$ (16)

Hence <u>by uniqueness</u> <u>this must be the result of a formal Laurent expansion</u>. Hence $\boxed{b_1(z=0) = 2}$ (17)

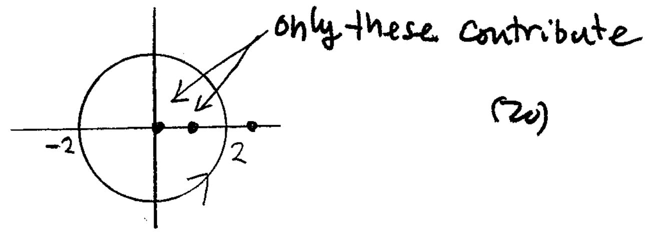
Similarly in the vicinity of $z=1$ we can write

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{(5z-2)/z}{z-1} \xrightarrow{z=1} \frac{3}{z-1} \Rightarrow \boxed{b_1(z=1)=3} \qquad (18)$$

Hence via the residue theorem: $\boxed{\oint_C dz\, f(z) = 2\pi i\,(2+3) = 10\pi i}$ (19)

Comment: There would have been the same number of residues had we considered the function

$$f(z) = \frac{5z-2}{z(z-1)(z-3)} \qquad (20)$$



integrated along the same contour $|z|=2$. However, the residues at the two poles would have been different.

[3] $$I = \oint_C dz \frac{(5z-2)}{z(z-1)^3} \qquad C = \text{circle } |z|=2 \qquad (21)$$

By the residue theorem we have $\oint_C \cdots = \underbrace{\oint_{C_0} \cdots}_{z=0 \text{ pole}} + \underbrace{\oint_{C_1} \cdots}_{z=1 \text{ pole}}$ (22)

$$\therefore \oint_C \cdots = \oint_{C_0} dz \frac{(5z-2)/(z-1)^3}{z} + \oint_{C_1} dz \frac{(5z-2)/z}{(z-1)^3} \qquad (23)$$

$$\oint_{C_0} \cdots = \oint_{C_0} dz \frac{(5z-2)/(z-1)^3}{z-0} = 2\pi i \left[b_1(z=0) = (-2)/(-1)^3 \right] = 4\pi i \qquad (24)$$

$$\oint_{C_1} dz \cdots = \oint_{C_1} dz \left\{ \frac{(5z-2)/z}{(z-1)^3} \right\} \tag{25}$$

<u>Comment:</u> At this stage we could expand the entire integrand in a Laurent series and pick out the coefficient $b_1(z=1)$. However, by virtue of the same uniqueness argument that we gave on p. CV-83 we can use the Cauchy derivative formula, which is a more convenient way to get the same result. Recall:

$$\boxed{\oint_C dz\, g(z) = \oint_C dz \frac{f(z)}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0)} \tag{26}$$

In (25) & (26)
$$\left.\begin{aligned} f(z) &= (5z-2)/z \\ &= (5-2/z) \\ n &= 2 \end{aligned}\right\} \Rightarrow \begin{aligned} f'(z) &= \frac{d}{dz}\left(5-\frac{2}{z}\right) = \frac{2}{z^2} \\ f''(z) &= -\frac{4}{z^3} \end{aligned} \tag{27}$$

Hence
$$\oint_{C_1} dz \cdots = \frac{2\pi i}{2!}\left[-\frac{4}{z^3}\right]_{z=1} = -4\pi i \tag{28}$$

Combining Eqs. (24) & (28) we see that

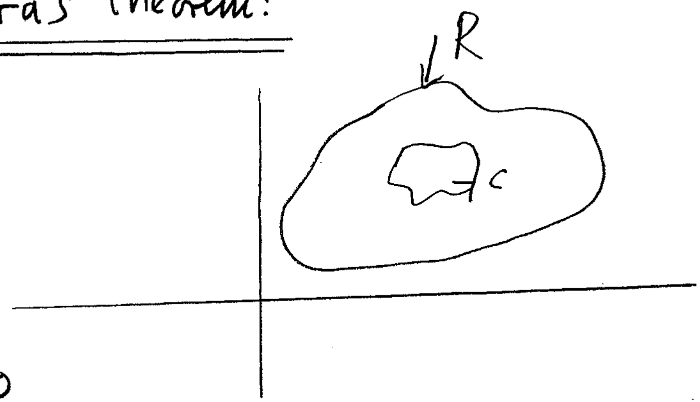
$$\oint_{C \Rightarrow |z|=2} dz \frac{(5z-2)}{z(z-1)^3} = \oint_{C_0} dz \cdots + \oint_{C_1} dz \cdots = \Big[\underbrace{+4\pi i}_{z=0} \; \underbrace{-4\pi i}_{z=1}\Big] = 0$$

This is an interesting example because it shows that a function can be non-analytic (has poles) and yet give $\oint_C dz \cdots = 0$ for some contour C, by <u>virtue of cancellations.</u>

See CV-45

Connection to Morera's Theorem:

Cauchy's Theorem



If $f(z)$ is analytic in R then $\oint_C dz\, f(z) = 0$

for any C.

Morera's Theorem (Converse of Cauchy theorem)

If $f(z)$ is continuous in R and if $\oint_C dz\, f(z) = 0$ <ins>for any</ins> C* in R then $f(z)$ is analytic in R.

In the previous example $f(z) = \dfrac{5z-2}{z(z-1)^3}$ is clearly not analytic in the region defined by the circle $|z| = 2$. Nonetheless $\oint dz\, f(z) = 0$. This does not contradict Morera's theorem, which holds that $f(z)$ is analytic only if $\oint_C dz\, f(z)$ *<ins>for any</ins> c! In the present example $\oint dz\, f(z) \neq 0$ if C is the circle $|z| = 0.5$

So the conditions for Morera's theorem are not met.

Two Views on Evaluating Integrals with Higher-Order Poles

Return to the previous problem:

$$I = \oint_C dz \, \frac{5z-2}{z(z-1)^3} \qquad C = \text{circle } |z| = 2 \qquad (1)$$

Residue theorem: $I = \oint_{C_0} dz \cdots + \oint_{C_1} dz \cdots$ (2)

Since $\oint_{C_1} dz \ldots$ involves a higher-order ($N=3$) pole we previously used the Cauchy formula:

$$\frac{1}{2\pi i}\oint_{C_0} dz\, g(z) = \frac{1}{2\pi i}\oint_{C_0} dz \frac{f(z)}{(z-z_0)^{n+1}} = \frac{1}{n!} f^{(n)}(z_0) \qquad (3)$$

This then gives:

$$\boxed{\oint_{C_0} dz \frac{f(z)}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0)} \qquad (4)$$

However, we have previously shown [p. CV-81,82] that we can evaluate the integral in (3) by writing

$$\boxed{\frac{1}{2\pi i}\oint_{C_0} dz\, g(z) = b_1 = \text{coefficient of } \left(1/z-z_0\right) \text{ in the Laurent expansion of } g(z)} \qquad (5)$$

Q: What is the connection between the use of (4) and (5) to evaluate $g(z)$?

A: When $g(z) = f(z)/(z-z_0)^{n+1}$, with $f(z)$ analytic at z_0 we can expand $f(z)$ as follows:

$$f(z) = f(z_0) + (z-z_0) f^{(1)}(z_0) + \frac{1}{2!}(z-z_0)^2 f^{(2)}(z_0) + \cdots + (z-z_0)^{n-1}\frac{1}{(n-1)!} f^{(n-1)}(z_0)$$
$$+ (z-z_0)^n \frac{1}{n!} f^{(n)}(z_0) + \cdots \qquad (6)$$

It follows from (6) that the integrand in (5) is given by

$$g(z) = \frac{f(z)}{(z-z_0)^{n+1}} = \frac{f(z_0)}{(z-z_0)^{n+1}} + \frac{f^{(1)}(z_0)}{(z-z_0)^{n}} + \frac{1}{2!}\,\frac{f^{(2)}(z_0)}{(z-z_0)^{n-1}} + \cdots$$
$$+ \frac{1}{(z-z_0)}\left[\frac{1}{n!}\,f^{(n)}(z_0)\right] + \frac{1}{(n+1)!}\,f^{(n+1)}(z_0) + \cdots \tag{7}$$

Now the general result that the residue of $g(z)$ at z_0 is just the coefficient b_1 of $1/(z-z_0)$ then gives

$$\boxed{b_1 = \left[\frac{1}{n!}\,f^{(n)}(z_0)\right] \Rightarrow \oint_{C_0} dz\, g(z) = 2\pi i\, b_1 = \frac{2\pi i}{n!}\,f^{(n)}(z_0)} \tag{8}$$

We thus the result in Eq. (4) obtained previously using the Cauchy formula.

<u>CONCLUSION:</u> When evaluating $\oint_{C_0} dz\, g(z) = \oint_{C_0} dz \frac{f(z)}{(z-z_0)^{n+1}}$

<u>We can use the Cauchy formula in (4) <u>directly</u></u> since this is equivalent to doing a Laurent expansion of $g(z)$ and finding the coefficient b_1 of $1/(z-z_0)$.

Evaluation of Real Integrals via Contour Integration

(A) Let R be a rational function of $\sin\theta, \cos\theta, \sin^2\theta, \cos^2\theta, \ldots$

For example:

$$R(\cos\theta, \sin\theta) \equiv R = \frac{a_1\cos\theta + a_2\cos^2\theta + \ldots + a_n\cos^n\theta + b_1\sin\theta + b_2\sin^2\theta + \ldots}{c_1\cos\theta + c_2\cos^2\theta + \ldots + c_n\cos^n\theta + d_1\sin\theta + d_2\sin^2\theta + \ldots} \tag{1}$$

By using contour integration we can evaluate integrals of the form

$$I = \int_0^{2\pi} d\theta \; R(\cos\theta, \sin\theta) \tag{2}$$

<u>Method:</u> Let $\boxed{z = 1e^{i\theta}} \Rightarrow dz = ie^{i\theta}d\theta = iz\,d\theta \Rightarrow \boxed{d\theta = -i\frac{dz}{z}}$ (3)

Then: $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

(4)

$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

Then:

$$\boxed{\int_0^{2\pi} d\theta \, R(\cos\theta, \sin\theta) = \oint_{C = \text{Unit Circle}} \left(-i\frac{dz}{z}\right) R\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right]} \tag{5}$$

<u>Example:</u>

$$I = \int_0^{2\pi} d\theta \frac{1}{a + \cos\theta} \qquad a > 1 \tag{6}$$

Note to begin that this is a real integral, which could in principle be evaluated using real integration. Nonetheless it is advantageous to carry out this integral via contour integration using (5).

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Combining (3) − (6) we have

$$I = -i \oint_{|z|=1} \frac{dz}{z} \frac{1}{a + \frac{1}{2}(z + \frac{1}{z})} = -i \oint dz \frac{2}{z^2 + 2az + 1} \qquad (7)$$

The poles will occur at the roots of $z^2 + 2az + 1 = 0$ which are given by

$$\alpha = -a + \sqrt{a^2 - 1} \qquad \beta = -a - \sqrt{a^2 - 1} \qquad (8)$$

$$\therefore I = -2i \oint_{|z|=1} dz \frac{1}{(z-\alpha)(z-\beta)} \qquad (9)$$

To evaluate I we must determine which of these poles (if either!) lies within the contour $|z| = 1$. Since $a > 1$ it follows immediately that $\beta < -1 \Rightarrow |\beta| > 1$. Hence the pole at $z = \beta$ lies outside the contour $|z| = 1$, and thus does not contribute to the integral.

For α we can show that this pole does fall within $|z| = 1$. To see this

$$a > 1 \Rightarrow 2(1+a) = 1 + (1+2a) > 0 \Rightarrow 1 + 2a > -1 \qquad (10)$$

Adding a^2 to both sides: $1 + 2a + a^2 > -1 + a^2 = a^2 - 1 \qquad (11)$

Take $\sqrt{}$ of both sides of (10) $\Rightarrow \underbrace{\sqrt{(1 + 2a + a^2)}}_{1+a} > \sqrt{a^2 - 1} \qquad (12)$

So $1 + a > \sqrt{a^2 - 1} \Rightarrow 1 > \underbrace{-a + \sqrt{a^2 \pm 1}}_{\alpha} \qquad (13)$

$\therefore$ Finally, $\boxed{|\alpha| < 1}$ → to ensure that $z = \alpha$ was inside the unit circle we took $a > 1$.

This can also be seen by noting that for $a > 1 \Rightarrow 1 < a < \infty$
$\alpha = -a + \sqrt{a^2 - 1}$ varies between $\alpha = 0$ $(a = \infty)$ and $\alpha = -1$ $(a = 1)$. Hence α must be in the range $0 < \alpha < -1 \Rightarrow |\alpha| < 1$.