Collecting together the coefficients of like powers of t gives:

$$(1-2tx+t^2)^{-1/2} = 1 \cdot P_0 + t P_1(x) + t^2 P_2(x) + t^3 P_3(x) + \ldots \tag{7}$$

$$= 1 + tx - \frac{1}{2}t^2 - \frac{3}{2}t^3 x + \frac{3}{8}t^4 + \frac{3}{2}t^2x^2 + \frac{5}{2}t^3x^3 + \ldots \tag{8}$$

$$= \underbrace{1}_{P_0(x)} + t\underbrace{[x]}_{P_1(x)} + t^2\underbrace{\left[\tfrac{3}{2}x^2 - \tfrac{1}{2}\right]}_{P_2(x)} + t^3\underbrace{\left[\tfrac{5}{2}x^3 - \tfrac{3}{2}x\right]}_{P_3(x)} + \ldots \tag{9}$$

These are the standard "textbook" expressions for $P_0(x), \ldots, P_3(x)$. Note that they agree with the values of the $P_n(x)$ obtained from the G-S method up to a (trivial!) overall normalization constant.

Rodrigues' Formula for $P_n(x)$:

For various applications it is useful to have several expressions for the $P_n(x)$. Another way of deriving the $P_n(x)$ is via the Rodrigues formula:

$$\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n} \tag{10}$$

Checks: $P_0(x) = \frac{1}{2^0 0!} \cdot 1 \, (x^2-1)^0 = 1$ ✓

$P_1(x) = \frac{1}{2 \cdot 1!}\frac{d}{dx}(x^2-1)^1 = \frac{1}{2} \cdot 2x = x$ ✓

$P_2(x) = \frac{1}{2^2 \cdot 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) = \frac{1}{8}\frac{d}{dx}(4x^3-4x)$
$= \frac{1}{2}(3x^2-1)$ ✓

Proof of Rodrigues' Formula:

Outline: If we accept the theorem that in any interval there is a unique set of orthogonal polynomials, then if we can show that the Rodrigues formula leads to an orthogonal set of polynomials in $[-1,1]$ then they must be the $P_n(x)$. Specifically we show that the formula leads to

$$\int_{-1}^{1} dx\, P_n(x) P_m(x) = \frac{2}{2n+1}\,\delta_{nm} \qquad (1)$$

Assume to start $n > m$; $d^n \equiv d^n/dx^n$

$$I = \int_{-1}^{1} dx\, P_n(x) P_m(x) = \int_{-1}^{1} dx \left[\frac{1}{2^n n!}\, d^n (x^2-1)^n\right]\left[\frac{1}{2^m m!}\, d^m (x^2-1)^m\right] \stackrel{?}{=} 0 \quad (n \neq m) \qquad (2)$$

Dropping constants, $$I = \int_{-1}^{1} dx\, \underbrace{\left[d^n (x^2-1)^n\right]}_{dv}\,\underbrace{\left[d^m (x^2-1)^m\right]}_{u} = \qquad (3)$$

$$\underbrace{\left[d^{n-1}(x^2-1)^n\right]}_{v}\,\underbrace{\left[d^m (x^2-1)^m\right]}_{u} - \int_{-1}^{1} dx\, \underbrace{\left[d^{n-1}(x^2-1)^n\right]}_{v}\,\underbrace{\left[d^{m+1}(x^2-1)^m\right]}_{du} \qquad (4)$$

→ polynomial ⊗ (x^2-1) → but ∃ one more power of (x^2-1) than there are derivatives on it ⇒ at the end we are left with a factor (x^2-1) which vanishes at $x = \pm 1$.

Continuing in this manner we see that

$$\int_{-1}^{1} dx \left[d^n (x^2-1)^n\right]\left[d^m (x^2-1)^m\right] = (-1)^n \int_{-1}^{1} dx\, (x^2-1)^n \left[d^{m+n} (x^2-1)^m\right] \qquad (5)$$

$$= 0 \quad \text{since } n > m \Rightarrow m+n > 2m \Rightarrow d^{m+n}(x^2-1)^m \equiv 0. \checkmark \qquad (6)$$

$$\therefore \boxed{I = 0 \quad \text{when } n \neq m} \qquad (7)$$

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To complete the proof of uniqueness we consider the case $m=n$. Reinstating the constants we get:

$$\int_{-1}^{1} dx\, P_n(x)\, P_n(x) = \frac{1}{(2^n n!)^2} (-1)^n \int_{-1}^{1} dx\, (x^2-1)^n \Big[d^{2n} \underbrace{(x^2-1)^n}_{\text{polynomial} \sim x^{2n}} \Big] \qquad (8)$$

When d^{2n} acts on x^{2n} there will be one term which does survive: All lower powers will be absent as a result of differentiation. This surviving term is given by

$$d^{2n} x^{2n} = d^{2n-1} (2n x^{2n-1}) = d^{2n-2} \left[(2n)(2n-1)\, x^{2n-2} \right] = \dots (2n)! \qquad (9)$$

Hence altogether $$\int_{-1}^{1} dx\, [P_n(x)]^2 = \frac{1}{2^{2n}(n!)^2} \cdot (2n)!\, \underbrace{(-1)^n \int_{-1}^{1} dx\, (x^2-1)^n}_{\frac{\sqrt{\pi}\; n!\; (-1)^n}{(n+\frac{1}{2})!}} \qquad (10)$$

} Jahnke-Emde p.20

$$(n+\tfrac{1}{2})! = \frac{\sqrt{\pi}\;\; 1\cdot 3\cdot 5 \dots (2n+1)}{2^{n+1}} \qquad (11)$$

} Jahnke-Emde p.11

Collecting these results together we find

$$\int_{-1}^{1} dx\, [P_n(x)]^2 = \frac{1}{2^{n-1}} \; \frac{(2n)!}{n!} \; \frac{1}{1\cdot 3\cdot 5 \dots (2n+1)} \qquad (12)$$

$$(2n)! = (2n)(2n-1)(2n-2)\dots 3\cdot 2\cdot 1 = \left[(2n)(2n-2)(2n-4)\dots \right] \left[(2n-1)(2n-3) \dots 3\cdot 1 \right]$$

$$\downarrow \quad (2\cdot n)\, 2(n-1)\;\; 2(n-2) \dots$$

$$= 2^n n! \qquad (13)$$

Hence $(2n)! = 2^n n! \left[(2n-1)(2n-3) \dots 3\cdot 1 \right]$ (14)

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$$\therefore \int_{-1}^{1} dx\, [P_n(x)]^2 = \frac{1}{2^{n-1}} \frac{[2^n \cancel{n!}]\,[\cancel{(2n-1)(2n-3)\cdots 3\cdot 1}]}{\cancel{n!}} \frac{1}{[\cancel{1\cdot 3\cdots(2n-1)}](2n+1)} \quad (15)$$

Hence finally
$$\boxed{\int_{-1}^{1} dx\, [P_n(x)]^2 = \frac{2^n}{2^{n-1}} \frac{1}{(2n+1)} = \frac{2}{2n+1}} \quad (16)$$

This establishes that the Rodrigues formula produces polynomials in $[-1,1]$ which have the same normalization & orthogonality properties as $P_n(x) = \sqrt{\frac{2}{2n+1}}\, \bar{P}_n(x)$, which is what we want.

[Recall that $\int_{-1}^{1} dx\, \bar{P}_n(x)\, \bar{P}_m(x) = \delta_{mn}$] (17)

Return to Normalization Questions:

The $\bar{P}_n(x)$ have a simple normalization as in (17), but $P_n(x)$ have another simple property:

$$\boxed{P_n(x) \xrightarrow{x=1} 1} \quad (18)$$

2 Proofs of (18):

(a) Generating function: $(1-2tx+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$ (19)

Set $x=1 \Rightarrow (1-2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(1) = 1\cdot P_0(1) + t\,P_1(1) + t^2 P_2(1) + \ldots$ (20)

$$\frac{1}{\sqrt{(1-t)^2}} = \frac{1}{1-t} = 1+t+t^2+\ldots \Rightarrow P_n(1) = 1 \checkmark \quad (21)$$

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(b) Rodrigues Formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[\underbrace{(x^2-1)}_{(x+1)(x-1)}]^n \quad \leftarrow \text{Differentiate each factor separately:} \qquad (22)$$

$$P_n(x) = \frac{1}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}\left\{ n(x-1)^{n-1}\cdot(x+1)^n + (x-1)^n\, n(x+1)^{n-1} \right\} \qquad (23)$$

Among the many terms which survive the differentiations, the only one which will survive in the very end when we set $x=1$ is this one since the differentiations eventually remove the factor $(x-1)$ completely. Thus at the end this term gives

$$P_n(x) = \frac{1}{2^n n!}\left\{ n!\,(x-1)^0(x+1)^n + \ldots \right\} \stackrel{x=1}{=} \frac{1}{2^n n!}\left\{ n!\, 1\cdot 2^n \right\} = 1 \checkmark \qquad (24)$$

THE LEGENDRE EQUATION

This is usually the starting point, since this equation can be obtained by solving the angular part of $\nabla^2\phi(\vec{x}) = 0$.

Here we reverse the process, by deriving the Legendre equation: We show that the polynomials defined by the Rodrigues formula [and which are unique for $-1 \leq x \leq +1$] also solve the Legendre equation

$$\boxed{(x^2-1)P_n''(x) + 2xP_n'(x) - n(n+1)P_n(x) = 0} \qquad (1)$$

Begin with the identity:

$$(x^2-1)\frac{d}{dx}(x^2-1)^n \equiv (x^2-1)\,d\,(x^2-1)^n = (x^2-1)\,n\,(x^2-1)^{n-1}2x = 2nx\,(x^2-1)^n \qquad (2)$$

Differentiate both sides w.r.t. x $(n+1)$ times: using the "binomial expansion" of the product rule:

$$d^m(uv) = u\,d^m v + m\,(du)\,(d^{m-1}v) + \frac{m(m-1)}{2!}(d^2u)(d^{m-2}v) + \cdots \qquad (3)$$

$$+ \frac{m!}{(m-k)!\,k!}(d^k u)(d^{m-k}v) + \cdots + (d^m u)\,v$$

Take $(n+1)$ derivatives of the l.h.s. of (2) [so that $m \to n+1$]

$$d^{n+1}\Big[\underbrace{(x^2-1)}_{u}\;\underbrace{d(x^2-1)^n}_{v}\Big] = (x^2-1)\,d^{n+2}(x^2-1)^n + (n+1)\Big[\underbrace{d(x^2-1)}_{2x}\Big]d^{n+1}(x^2-1)^n \qquad (4)$$

$$+ \frac{(n+1)(n+1-1)}{2!}\Big[\underbrace{d^2(x^2-1)}_{2}\Big]\Big[d^n(x^2-1)^n\Big] \qquad (5)$$

$$+ \cdots \otimes \Big[\underbrace{d^3(x^2-1)}_{0}\Big]\Big[d^{n-1}(x^2-1)^n\Big] + \cdots$$

$\leftarrow$ also for remaining terms

Collecting terms d^{n+1} (l.h.s. of (2)) $\Rightarrow$

$$\rightarrow\; d^{n+1}\big[(x^2-1)\,d(x^2-1)^n\big] = (x^2-1)\,d^{n+2}(x^2-1)^n + (n+1)\,2x\,d^{n+1}(x^2-1)^n \qquad (6)$$

$$+ n(n+1)\,d^n(x^2-1)^n$$

Next evaluate d^{n+1} acting on the r.h.s. of (2):

$$d^{n+1}\Big[\underbrace{2nx}_{u}\,\underbrace{(x^2-1)^n}_{v}\Big] = 2nx\left[d^{n+1}(x^2-1)^n\right] + (n+1)\left[d(2nx)\right]\left[d^n(x^2-1)^n\right]$$

$$+\frac{n(n-1)}{2!}\underbrace{\left[d^2(2nx)\right]}_{=0}\left[d^{n-1}(x^2-1)^n\right] + \ldots \qquad (7)$$

0 ← also for remaining terms

→ Hence $d^{n+1}\left[2nx(x^2-1)^n\right] = 2nx\left[d^{n+1}(x^2-1)^n\right] + 2n(n+1)\left[d^n(x^2-1)^n\right]$ (8)

Subtracting (8) from (6) we get

$$d^{n+1}\left[(x^2-1)\,d(x^2-1)^n\right] - d^{n+1}\left[2nx(x^2-1)^n\right] \equiv 0 \qquad (9)$$

$$= (x^2-1)\,d^{n+2}(x^2-1)^n + \Big[\overbrace{(n+1)2x - 2nx}^{2x}\Big]d^{n+1}(x^2-1)^n + \Big[\overbrace{n(n+1) - 2n(n+1)}^{-n(n+1)}\Big]d^n(x^2-1)^n \qquad (10)$$

Hence $0 = \underbrace{(x^2-1)\,d^{n+2}(x^2-1)^n}_{d^2[2^n n!\,P_n]} + 2x\,\underbrace{d^{n+1}(x^2-1)^n}_{d[2^n n!\,P_n]} - n(n+1)\,\underbrace{d^n(x^2-1)^n}_{2^n n!\,P_n}$ (11)

Finally: $\boxed{(x^2-1)\,P_n''(x) + 2x\,P_n'(x) - n(n+1)\,P_n(x) = 0}$ (12)

Sturm–Liouville Form of Legendre Equation

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$$\frac{d}{dx}\left[(x^2-1)\frac{dP_n(x)}{dx}\right]-n(n+1)P_n(x)=0 \qquad (1)$$

$$\Rightarrow \quad (x^2-1)\frac{d^2P_n(x)}{dx^2}+\underbrace{(2x)}\frac{dP_n}{dx}-n(n+1)\,P_n(x)=0 \checkmark \qquad (2)$$

This can be used for an alternative proof that $\langle P_n | P_m\rangle = 0$:

(1) $\Rightarrow$

$$\frac{d}{dx}\left[(x^2-1)\,dP_n/dx\right]-n(n+1)P_n(x)=0 \quad \leftarrow \text{multiply by } P_m(x) \qquad (3a)$$

also $$\frac{d}{dx}\left[(x^2-1)\,dP_m/dx\right]-m(m+1)P_m(x)=0 \quad \leftarrow \text{multiply by } P_n(x) \qquad (3b)$$

After multiplying subtract (3b) from (3a) $\Rightarrow$ (after integrating)

$$0=\int_{-1}^{1}dx\left\{\left[\underbrace{P_m\frac{d}{dx}\left[(x^2-1)\frac{dP_n}{dx}\right]-n(n+1)P_mP_n}_{\neq 0}\right]-\left[\underbrace{P_n\frac{d}{dx}\left[(x^2-1)\frac{dP_m}{dx}\right]-m(m+1)P_nP_m}_{\neq 0}\right]\right\} \qquad (4)$$

$$0=\left[m(m+1)-n(n+1)\right]\int_{-1}^{1}dx\,P_n(x)P_m(x)+\int_{-1}^{1}dx\left\{\underset{u}{P_m}\underbrace{\frac{d}{dx}\left[(x^2-1)\frac{dP_n}{dx}\right]}_{dv}-P_n\frac{d}{dx}\left[(x^2-1)\frac{dP_m}{dx}\right]\right\} \qquad (5)$$

In (5):

$$\text{Second}\int_{-1}^{1}dx = \underbrace{P_m(x^2-1)\frac{dP_n}{dx}\Big|_{-1}^{1}}_{=0}-\underbrace{(n\leftrightarrow m)}_{=0}-\int_{-1}^{1}dx\left\{\cancel{\frac{dP_m}{dx}(x^2-1)\frac{dP_n}{dx}}-\cancel{\frac{dP_n}{dx}(x^2-1)\frac{dP_m}{dx}}\right\} \qquad (6)$$

(cancel)

Hence in (5) $$0=\left[m(m+1)-n(n+1)\right]\int_{-1}^{1}dx\,P_n(x)P_m(x)$$

$$\therefore \quad 0=\left[m(m+1)-n(n+1)\right]\langle P_n | P_m\rangle$$

$$\therefore \quad m\neq n \Rightarrow \langle P_n | P_m\rangle = 0 \checkmark$$