Adjoint Continued:

4) $(A^\dagger)^\dagger = A$

def. $A^\dagger$ — def. of [redacted] $*$ — def of $\dagger$

Proof: $\langle x | (A^\dagger)^\dagger y \rangle \overset{\text{def. } A^\dagger}{=} \langle A^\dagger x | y \rangle = \langle y | A^\dagger x \rangle^* = \langle Ay | x \rangle^* = \langle x | Ay \rangle$

$\therefore (A^\dagger)^\dagger = A$

5) $(AB)^\dagger = B^\dagger A^\dagger$

Proof: $\langle x | (AB)^\dagger y \rangle \overset{\text{def. } \dagger}{=} \langle AB x | y \rangle \equiv \langle A(Bx) | y \rangle = \langle Bx | A^\dagger y \rangle$

$= \langle x | B^\dagger A^\dagger y \rangle \Rightarrow (AB)^\dagger = B^\dagger A^\dagger$

Theorem: A is a l.f. on V. Then $A = 0 \Leftrightarrow \langle x | Ay \rangle = 0 \ \forall x, y$.

Proof: a) $A = 0 \Rightarrow \langle x | Ay \rangle = 0$ trivial

b) If $\langle x | Ay \rangle = 0 \ \forall x, y$, let $x = Ay \Rightarrow \langle Ay | Ay \rangle = 0$

But by the axioms for an inner product space $\Rightarrow |Ay\rangle = 0$

But if $Ay = 0$ for all $y \Rightarrow$ $\underline{A = 0}$ ✓

Theorem: Given a $\left\{ \begin{array}{l} \text{self adjoint} \\ \text{arbitrary} \end{array} \right\}$ l.f. A on V where $V = \left\{ \begin{array}{l} \text{inner product} \\ \text{unitary} \end{array} \right\}$ space then $A = 0 \Leftrightarrow \langle x | Ax \rangle = 0$.

Proof: not proved in class.

Theorem: If $A = A^\dagger$ & $B = B^\dagger$ then $AB = (AB)^\dagger$ or $BA = (BA)^\dagger$ only if $[A, B] = 0$.

Proof: $(AB)^\dagger = B^\dagger A^\dagger \left. \begin{array}{l} = BA \\ \equiv AB \end{array} \right\}$ only if $[A, B] = 0$

<u>Definition:</u> If $A^{\dagger} = -A$ A is skew-symmetric or skew-Hermitian

<u>Note:</u> $A = \underbrace{\frac{1}{2}(A + A^{\dagger})}_{\text{self-adjoint}} + \underbrace{\frac{1}{2}(A - A^{\dagger})}_{\text{skew self-adjoint}}$

✪ <u>Theorem:</u> A is a l.t. on V ; <u>A is Hermitian iff. $\langle x|Ax\rangle$ = real</u>

<u>Proof:</u> a) If $A = A^{\dagger} \Rightarrow \langle x|Ax\rangle \underset{\text{def. } A^{\dagger}}{=} \langle A^{\dagger}x|x\rangle \underset{\text{def. } *}{=} \langle x|A^{\dagger}x\rangle^{*} \underset{A=A^{\dagger}}{=} \langle x|Ax\rangle^{*}$

$\therefore A = A^{\dagger} \Rightarrow \langle x|Ax\rangle$ is real.

b) If $\langle x|Ax\rangle$ is always real $(\forall x) \Rightarrow \langle x|Ax\rangle \underset{\text{assumed}}{=} \langle x|Ax\rangle^{*} \underset{\text{def. } *}{=} \langle Ax|x\rangle \underset{\text{def. } \dagger}{=} \langle x|A^{\dagger}x\rangle$

$\therefore \langle x|Ax\rangle = \langle x|Ax\rangle^{*} \Rightarrow A = A^{\dagger}$

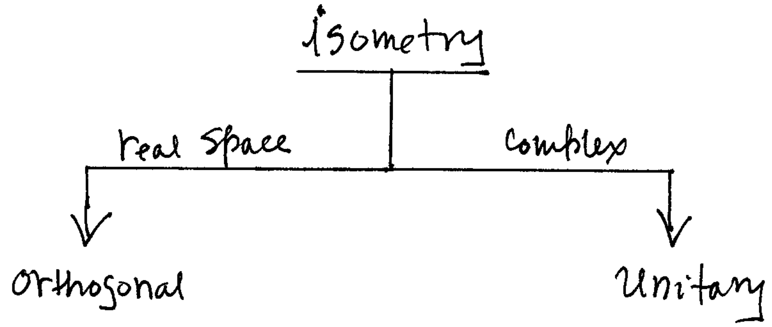
<u>Notation:</u> $\langle x|Ax\rangle \equiv$ expectation value of A in the state x.

In QM $\langle x|Ax\rangle \equiv \langle x|A|x\rangle$ is the value obtained by measuring the eigenvalue of A. Since this is a physical quantity $\langle x|Ax\rangle$ must be real. Hence, observables in QM are the (real) eigenvalues of Hermitian operators.

ISOMETRIES: ORTHOGONAL & UNITARY TRANSFORMATIONS

Definition: A is an isometry if $A^{\dagger} = A^{-1}$ (iso = same -metry = measure)

$A^{\dagger} = A^{-1} \Rightarrow AA^{\dagger} = 1 = A^{\dagger}A$



SUMMARY OF TRANSFORMATIONS

$(A)_{ij} = \alpha_{ij}$

	Self Adjoint Transf. ($A^{\dagger} = A$)	Isometric Transf. ($A^{\dagger} = A^{-1}$)
Unitary (complex) Space	Hermitian [redacted] $(A^{\dagger})_{ij} = \alpha_{ji}^{*}$	Unitary $\sum_k \alpha_{ik}\alpha_{jk}^{*} = \delta_{ij}$
Euclidean (real) Space	Symmetric $(A^{\dagger})_{ij} = \alpha_{ji}$	Orthogonal $\sum_k \alpha_{ik}\alpha_{jk} = \delta_{ij}$

Theorem on Isometric Transformations:

The following 3 conditions are equivalent statements about unitary transformations:

a) $U^{\dagger}U = I$ (or $U^{\dagger} = U^{-1}$)

b) $\langle Ux | Uy \rangle = \langle x | y \rangle$

c) $|Ux| = |x|$

Unitary Transformations (cont'd):

Proof: We show (1) ⇒ (2) ⇒ (3) ⇒ (1)

(1) ⇒ (2) $\quad \mathcal{U}^{+}\mathcal{U} = I \Rightarrow \langle \mathcal{U}x | \mathcal{U}y \rangle \underset{\uparrow \text{def } +}{=} \langle x | \underbrace{\mathcal{U}^{+}\mathcal{U}}_{I} y \rangle = \langle x | y \rangle$ ✓ (1)

(2) ⇒ (3) $\quad \langle \mathcal{U}x | Uy \rangle = \langle x | y \rangle \underset{y=x}{\Rightarrow} \underbrace{\langle \mathcal{U}x | \mathcal{U}x \rangle}_{|Ux|^2} = \underbrace{\langle x | x \rangle}_{|x|^2} \Rightarrow |\mathcal{U}x|^2 = |x|^2$ (*)
$\Rightarrow |\mathcal{U}x| = |x|$ ✓ (2)

(3) ⇒ (1) $\quad \left. |\mathcal{U}x|^2 = \langle \mathcal{U}x | Ux \rangle = \langle x | U^{+}\mathcal{U}x \rangle \right\} \Rightarrow U^{+}U = I$ (3)
$\hookrightarrow = \langle x | x \rangle$

Isometric Transformations preserve the lengths of vectors, (*) and scalar products, and hence they also preserve the angles between vectors:
In an abstract (i.e. any) vector space

$$\cos\theta \equiv \frac{\langle x|y\rangle}{|x|\cdot|y|} \xrightarrow{U} \frac{\langle Ux|Uy\rangle}{|Ux|\cdot|Uy|} = \frac{\langle x|y\rangle}{|x|\cdot|y|} \checkmark \quad (4)$$

Side Comment: If $\{x_i\}$ is a CON basis so is $\{Ux_i\}$.

Uniterity Constraints: $(U)_{ij} = u_{ij} \quad (U^{+})_{ij} = u^{*}_{ji} \Rightarrow$ (5)

$$\underset{\delta_{ik}}{\overset{\shortparallel}{(I)_{ik}}} = (UU^{+})_{ik} = \sum_j u_{ij}(U^{+})_{jk} = \sum_j u_{ij}u^{*}_{kj} \Bigg\} \boxed{\sum_j u_{ij}u^{*}_{kj} = \delta_{ik}} \quad (6)$$

↳ these are the familiar orthogonality relations when u_{ij} = real.

Check: 2x2 $\quad U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$i=1 \quad k=1 \Rightarrow \delta_{11} = 1 \overset{?}{=} \sum_j u_{1j}u^{*}_{1j} = u_{11}u^{*}_{11} + u_{12}u^{*}_{12} = \cos^2\theta + \sin^2\theta = 1$ ✓

$i=1 \quad k=2 \Rightarrow \delta_{12} = 0 \overset{?}{=} \sum_j u_{1j}u_{2j} = u_{11}u^{*}_{21} + u_{12}u^{*}_{22} = \cos\theta(-\sin\theta) + \sin\theta\cos\theta$
$= 0$ ✓

(7)

Theorem: If A is orthogonal then $\det A = \pm 1$

Proof: $AA^{+} = I \Rightarrow \det(AA^{+}) = (\det A)(\det A^{+}) = \det I = 1$ (8)

$\hookrightarrow = (\det A^{-1}) = \det A$

$\therefore (\det A)^2 = 1 \Rightarrow \boxed{\det A = \pm 1}$ (9)

Note that if A is orthogonal $\Rightarrow A^{+} = A^{T}$ and $\det A^{T} = \det A$ (Recall volume argument)

$\det A = +1 \Rightarrow$ rotation

$\det A = -1 \Rightarrow$ inversion

Theorem: If U is Unitary then $|\det U| = 1 \Leftrightarrow \det U = e^{i\theta}$

Proof: Assume U is diagonalized so that $U = \begin{pmatrix} u_{11} & & 0 \\ & u_{22} & \\ 0 & & u_{33} \end{pmatrix}$ (10)

$$U^{-1} = \begin{pmatrix} 1/u_{11} & & 0 \\ & 1/u_{22} & \\ 0 & & 1/u_{33} \end{pmatrix} = U^{+} \quad (11)$$

$$(\det U)(\det U^{+}) = (u_{11}u_{22}u_{33}) \frac{1}{u_{11}u_{22}u_{33}} = 1 \quad (12)$$

However, by definition the elements of U^{+} are $\begin{pmatrix} u_{11}^{*} & & 0 \\ & u_{22}^{*} & \\ 0 & & u_{33}^{*} \end{pmatrix}$ (13)

Hence $U^{+} = U^{-1} \Rightarrow u_{11}^{*} = 1/u_{11}$ etc $\Rightarrow |u_{11}|^2 = 1$

$\Rightarrow u_{11} = e^{i\theta_1}$ etc.

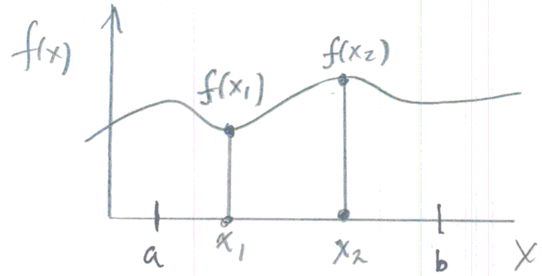
U must then have the form

$$U = \begin{pmatrix} e^{i\theta_1} & & 0 \\ & e^{i\theta_2} & \\ 0 & & e^{i\theta_3} \end{pmatrix} \Rightarrow \boxed{\begin{aligned} &\det U = e^{i(\theta_1+\theta_2+\theta_3)} \equiv e^{i\theta} \\ \Rightarrow\ &|\det U| = 1 \end{aligned}} \quad (14)$$

INTRODUCTION TO HILBERT SPACE

TWO (RELATED) VIEWS OF A FUNCTION AS A VECTOR IN AN ∞-DIM SPACE

(I)



In this way a function is represented in the interval $[a,b]$ by an ∞ set of values $(f(x_a), f(x_1), \ldots f(x_b))$ which give the "projections" of $f(x)$ on the "axes" $(x_a, x_1, \ldots x_b)$. This is similar to specifying the components of an n-dimensional vector by giving its components along the n-axes:

$\vec{V} = (V_1, V_2, \ldots V_n)$.

(II) We can also represent a function via a Taylor series in $[a,b]$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow f(x) = [a_0, a_1, \ldots a_n \ldots]$$

A function which is piecewise continuous (continuous in every subinterval in $[a,b]$)
These two are related: A function which is piecewise continuous can be expanded as in (II) via a Taylor series [More later!]

SCALAR PRODUCT: For 2 functions $f(x), g(x)$

$$\boxed{\langle f|g\rangle \equiv \int_a^b dx\, f^*(x)\, g(x)} \qquad (1)$$

NORM OF A FUNCTION $N[f] \equiv \langle f|f\rangle = \int_a^b dx\, |f(x)|^2 \qquad (2)$

Side Comment on Scalar Product:

The definition $\langle f|g\rangle = \int_a^b dx\, f^*(x) g(x)$ is the natural extension of the usual finite dimensional result:

$$\langle a|b\rangle = \vec{a}\cdot\vec{b} = \sum_{i=1}^{N} a_i^* b_i = a_1^* b_1 + a_2^* b_2 + \dots + a_N^* b_N$$

Let $a_1 \to a(x_1), \dots, a_N \to a(x_N)$; $b_1 \to b(x_1)$, $b_N \to b(x_N)$

Then $\langle a|b\rangle \to \sum_i a^*(x_i)\, b(x_i) \to \int dx\, a^*(x)\, b(x)$ ✓

The only additional ingredient needed is the interval over which the integral is evaluated.

HILBERT SPACE:

The set of functions for which the norm is finite constitutes Hilbert space. Sometimes to ensure that the norm is finite, especially when the integration limits are ∞, a weighting function $\omega(x)$ is introduced:

$$\boxed{\langle f|g\rangle \longrightarrow \int_a^b dx\, f^*(x)\, g(x)\, W(x)} \quad (3)$$

For example: $W(x) \sim e^{-x}$ (Laguerre) or $e^{-x^2/2}$ (Hermite)

Reisz–Fischer Theorem: (not proved here)

Functions with $N[f] < \infty$ are complete: A Hilbert space is then a complete linear vector space with a complex scalar product.

DIGRESSION ON "∞" [G.F. Cantor – Theory of Transfinite Numbers]

- Denumerable ∞: $\aleph_0$ ["Aleph-Null"] — 1, 2, 3, . . . ; 2, 4, 6, ...
- Continuum: C [points on a line]

KNOW CANTOR PROOF!! SEE "WORLD OF MATH" or GAMOW "ONE, TWO, THREE... INFINITY"

(I) $\Rightarrow$ Continuum infinity to specify points

(II) $\Rightarrow$ Denumerable infinity " " "

However, for a piecewise continuous function a denumerable ∞ is enough.

Separable Hilbert Space: One having at least one denumerable basis. Otherwise the Hilbert space is non separable:

NOTATION IN HILBERT SPACE

(a) $\langle f|g\rangle = 0 \Rightarrow f, g$ are orthogonal in $[a,b]$

(b) $\langle f|f\rangle = 1 \Rightarrow f$ is normalized in $[a,b]$

(c) $\langle f_i|f_j\rangle = \delta_{ij} \Rightarrow f_i, f_j$ are orthonormal in $[a,b]$

Example: $f_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx} \quad ; n = 0, \pm 1, \pm 2, \ldots$

The set $\{f_n(x)\}$ is orthonormal in $[-\pi, \pi]$. To see this:

$$\langle f_n|f_m\rangle = \int_{-\pi}^{\pi} f_n^*(x) f_m(x)\, dx = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\pi}^{\pi} (e^{inx})^* e^{imx}\, dx \qquad (1)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, e^{i(m-n)x} = \frac{1}{2\pi i(m-n)} e^{i(m-n)x}\Big]_{-\pi}^{\pi} = 0 \text{ if } m \neq n \qquad (2)$$

However, when $n = m$ (2) gives

$$\langle f_n|f_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx \cdot 1 = 1 \checkmark \qquad (3)$$

Hence altogether $\langle f_n|f_m\rangle = \delta_{mn}$ ✓

When a (non-negative) weight function $w(x)$ is used (as previously) then

$$\langle f_n|f_m\rangle \equiv \int_a^b dx\, f_n^*(x) f_m(x) w(x) = \delta_{mn} \Rightarrow \text{orthonormality}$$