[3] Idempotent Matrices

$\hookrightarrow A^2 = A$

Example: $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$; $A^2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ ✓

Similarly: $A^3 = A \cdot A^2 = A \cdot A = A^2 = A$ etc.

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ are also idempotent

[4] Non-Singular Matrices = Invertible Matrices:

A is non-singular if $\exists\, B \ni AB = I$ (recall p. F106/107)

If $\det A = 0$ it is singular (no inverse), e.g.

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

We will later show that in general if 2 rows or columns of a matrix are lin. dep. then A is singular.

[5] Transpose of a matrix $\equiv A^T = \tilde{A}$

$A = (\alpha_{ij}) \Rightarrow \quad A^T = \alpha_{ji}$ (rows $\Leftrightarrow$ columns)

a) $(A^T)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(AB)^T = B^T A^T \rightarrow$ Proof: $(A) = \alpha_{ij}$

$[A]_{ij} = \alpha_{ij}$ $\quad [B]_{ij} = \beta_{ij}$

Then $[A^T]_{ij} = \alpha_{ji}$; $[B^T]_{ij} = \beta_{ji}$

$$\left.\begin{aligned} [AB]_{ij} &= \sum_k \alpha_{ik}\beta_{kj} \Rightarrow [(AB)^T]_{ij} = \sum_k \alpha_{jk}\beta_{ki} \\ [B^T A^T]_{ij} &= \sum_k [B^T]_{ik}[A^T_{kj}] = \sum_k \beta_{ki}\alpha_{jk} = \sum_k \alpha_{jk}\beta_{ki} \end{aligned}\right\} \Rightarrow (AB)^T = B^T A^T$$

[6] Symmetric & Antisymmetric Matrices

$S = (a_{ij})$ is symmetric if $S = S^T \Rightarrow a_{ij} = a_{ji}$

$A = (a_{ij})$ is antisymmetric if $A = -A^T \Rightarrow a_{ij} = -a_{ji}$

in this case diagonal elements $\equiv 0 \Rightarrow \boxed{\mathrm{Tr}\, A = 0}$

Any square matrix can be decomposed into a sum of a symmetric and an antisymmetric matrix:

$$M = \underbrace{\tfrac{1}{2}(M + M^T)}_{\equiv S} + \underbrace{\tfrac{1}{2}(M - M^T)}_{\equiv A} \qquad (1)$$

For an $n \times n$ matrix there are n^2 entries in the original matrix M. These break up into

$$\underbrace{M}_{n^2} = \underbrace{S}_{\frac{1}{2}n(n+1)} + \underbrace{A}_{\frac{1}{2}n(n-1)} \qquad (2)$$

Example: $n=3$

$$\underbrace{M}_{3\times 3 = 9} = \underbrace{S}_{\frac{1}{2}(3\cdot 4) = 6} + \underbrace{A}_{\frac{1}{2}(3\cdot 2) = 3} \qquad (3)$$

The decomposition in (1) & (2) is useful because it allows some simplifications from identities such as:

$$\epsilon_{ijk}[A]_{jk} \neq 0 \quad \text{but} \quad \epsilon_{ijk}[S]_{jk} \equiv 0 \qquad (4)$$

where the sum is over all permutations P of the integers 1, 2, ...n and where the + or - sign is affixed to each product according to whether P is even or odd. F117

Examples:

1) 2 x 2 $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

P (1)	P (2)	Sign P
1	2	+
2	1	-

$\therefore$ det $A = +a_{11}\ a_{22} - a_{21}\ a_{12}$

2) 3 x 3

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

P(1)	P(2)	P(3)	Sign P
1	2	3	+
1	3	2	-
2	1	3	-
2	3	1	+
3	1	2	+
3	2	1	-

$$\therefore \det A = \begin{array}{l} +a_{11}\ a_{22}\ a_{33} \\ +a_{21}\ a_{32}\ a_{13} \\ +a_{31}\ a_{12}\ a_{23} \\ -a_{11}\ a_{32}\ a_{23} \\ -a_{21}\ a_{12}\ a_{33} \\ -a_{31}\ a_{22}\ a_{13} \end{array}$$

do this as an example

We now summarize some of the properties of the determinant function. These properties can all be proven from our definition but we will not go into the proofs. If these properties are not familiar satisfy yourself that they are valid for 2 x 2 and 3 x 3 matrices.

Determinants:

A function which acts on matrices $\rightarrow$ scalars

Definition:

$$\det A = \sum_{\substack{\text{permutations} \\ P}} \left\{ \pm\, a_{P(1)1}\, a_{P(2)2} \cdots a_{P(n)n} \right\}$$

(+ : even; − : odd permutation)

$$= \sum_{P} \left\{ \pm \prod_{i=1}^{n} a_{P(i)i} \right\}$$

Properties of Determinants:

1) A common factor of a row or column can be factored out.
2) $\det A = 0$ if any row or column $= 0$.
3) $\det A = 0$ if 2 rows or columns are lin. dependent.
4) $\det I = 1$
5) $\det A$ is unchanged if a scalar multiple of one row or column is added to another row or column [Useful in computations].
6) $\det A \rightarrow -\det A$ when 2 rows are interchanged (or 2 columns)
7) $\det A^T = \det A$

★ 8) $\det A = 0 \Leftrightarrow$ the row or column vectors are lin. dep.

The determinant as a volume:

$$\det \begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{pmatrix} = \begin{aligned} &A_x B_y C_z + B_x C_y A_z + C_x B_z A_y \\ &- A_z B_y C_x - B_z C_y A_x - C_z B_x A_y \end{aligned}$$

$$= A_x (B_y C_z - B_z C_y) + A_y (B_z C_x - B_x C_z) + A_z (B_x C_y - B_y C_x) = \vec{A} \cdot (\vec{B} \times \vec{C})$$

= Solid (parallelopiped) volume determined by $\vec{A}, \vec{B}, \vec{C}$

The geometric picture can be used as a mnemonic for the properties of a determinant:

Properties [from previous page]

1) 2) obvious ✓

3) $\vec{A}\cdot(\vec{B}\times\vec{C}) \rightarrow \vec{A}\cdot(\vec{B}\times\vec{B}) = 0$ etc. ✓

4) volume of unit cube $= 1$

5) $\vec{A}\cdot(\vec{B}\times\vec{C}) \rightarrow (\vec{A}+\lambda\vec{B})\cdot(\vec{B}\times\vec{C}) = \vec{A}\cdot(\vec{B}\times\vec{C}) + \lambda\,\underbrace{\vec{B}\cdot(\vec{B}\times\vec{C})}_{0}$ etc. ✓

6) $\vec{A}\cdot(\vec{B}\times\vec{C}) = -\vec{A}\cdot(\vec{C}\times\vec{B})$

7) The volume is the same whether A, B, C are rows or columns ✓

★ 8) a) If $\vec{C} = \lambda\vec{B}$ then $\vec{A}\cdot(\vec{B}\times\vec{C}) \rightarrow \lambda\vec{A}\cdot(\vec{B}\times\vec{B}) = 0$

Hence if 2 rows (or columns) are lin. dep $\Rightarrow$ volume $\rightarrow 0$

b) (Converse) if $\det A = 0$ this means that the vectors $\vec{A}, \vec{B}, \vec{C}$ have 0 3-dim volume. This means that these vectors must "live" in a lower dimensional space, such as a plane $(d=2)$ or a line $(d=1)$. But 3 vectors $\vec{A}, \vec{B}, \vec{C}$ in $d=2$ or $d=1$ must be lin. dependent.

$$\therefore \boxed{\det A = 0 \Rightarrow \vec{A}, \vec{B}, \vec{C} \text{ are lin. dependent}}$$

This geometric picture generalizes to higher dimension matrices & their determinants.

FINDING THE INVERSE OF A MATRIX:

F120

The "Cofactor" Rule for det A:

Definition: Let $A = (a_{ij})$. The cofactor $|A_{ij}|$ of a_{ij} is a number given by $(-1)^{i+j} \otimes$ det $(n-1)\times(n-1)$ matrix formed by deleting the row i and column j in A.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Rightarrow |A_{12}| = (-1)^{1+2}(a_{21}a_{33} - a_{23}a_{31})$$

$$= a_{23}a_{31} - a_{21}a_{33}$$

Cofactor Rule for det A:

$$\boxed{\begin{aligned} \det A &= \sum_{j=1}^{n} a_{ij}\,|A_{ij}| \quad \text{for a fixed } i \text{ (row)} \\ &= \sum_{i=1}^{n} a_{ij}\,|A_{ij}| \quad \text{for a fixed } j \text{ (column)} \end{aligned}}$$

Example: 3x3

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det A = a_{11}|A_{11}| + a_{12}|A_{12}| + a_{13}|A_{13}|$$

$$|A_{11}| = (-1)^{1+1}(a_{22}a_{33} - a_{23}a_{32}) \; ; \; |A_{12}| = (-1)^{1+2}(a_{21}a_{33} - a_{23}a_{31})$$

$$|A_{13}| = (-1)^{1+3}(a_{21}a_{32} - a_{22}a_{31})$$

$$\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

✓ ✓ F117

CONSTRUCTING THE INVERSE MATRIX:

We start with

$$\det A = \sum_{j=1}^{n} a_{ij} \, |A_{ij}| \qquad \text{(fixed row; sum over columns)} \qquad (1)$$

Form the matrix $\displaystyle\sum_{j=1}^{n} a_{ij} \, |A_{kj}|$ (2)

Claim: This matrix is just $= \delta_{ik} \det A$ (3)

Proof: For $i = k$ (2) = (3) obviously. For $i \neq k$ $\delta_{ik} \det A = 0$

So we have to prove that this is true for $\sum_{j=1} a_{ij} \, |A_{kj}|$. This can be done generally by showing that this yields the det of a matrix with 2 equal columns or rows. Here we simply illustrate this for 3x3:

$$\delta_{12} \det A = 0 \stackrel{?}{=} \sum_{j=1}^{n} a_{ij} |A_{kj}| = \sum_{j=1}^{n=3} a_{1j} \, |A_{2j}| = a_{11}|A_{21}| + a_{12}|A_{22}| + a_{13}|A_{23}| \qquad (4)$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Rightarrow \begin{aligned} |A_{21}| &= (-)(a_{12}a_{33} - a_{13}a_{32}) \\ |A_{22}| &= (+)(a_{11}a_{33} - a_{13}a_{31}) \\ |A_{23}| &= (-)(a_{11}a_{32} - a_{12}a_{31}) \end{aligned} \qquad (5)$$

Hence

$$\sum_{j=1}^{n=3} a_{1j} |A_{2j}| = (-)a_{11}(\cancel{a_{12}a_{33}} - \cancel{a_{13}a_{32}}) + a_{12}(\cancel{a_{11}a_{33}} - a_{13}a_{31}) - a_{13}(\cancel{a_{11}a_{32}} - \cancel{a_{12}a_{31}}) = 0 \qquad (6)$$

Hence we can write

$$\boxed{\delta_{ik} \det A = \sum_{j=1}^{n} a_{ij} \, |A_{kj}|} \qquad (7)$$

CONSTRUCTING THE INVERSE MATRIX:

We start with

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$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Rightarrow \begin{aligned} |A_{21}| &= (-)(a_{12}a_{33} - a_{13}a_{32}) \\ |A_{22}| &= (+)(a_{11}a_{33} - a_{13}a_{31}) \\ |A_{23}| &= (-)(a_{11}a_{32} - a_{12}a_{31}) \end{aligned} \qquad (5)$$

Hence
$$\sum_{j=1}^{n=3} a_{ij}\,|A_{2j}| = (-)a_{11}(\cancel{a_{12}a_{33}} - \cancel{a_{13}a_{32}}) + a_{12}(\cancel{a_{11}a_{33}} - a_{13}a_{31}) - a_{13}(\cancel{a_{11}a_{32}} - \cancel{a_{12}a_{31}}) = 0 \qquad (6)$$

Hence we can write

$$\boxed{\delta_{ik}\det A = \sum_{j=1}^{n} a_{ij}\,|A_{kj}|} \qquad (7)$$

To construct the inverse matrix we next introduce the classical adjoint matrix (to be distinguished later from the Hermitian adjoint. Classical adjoint of $A \equiv \mathrm{Adj}\, A$:

$$(\mathrm{Adj}\, A)_{ij} = |A_{ji}| \qquad \text{ex: } (\mathrm{Adj}\, A)_{12} = |A_{21}| \tag{8}$$

Then consider:

$$(A \cdot \mathrm{Adj}\, A)_{ij} = \sum_{k=1}^{n} a_{ik}\, (\mathrm{Adj}\, A)_{kj} = \sum_{k=1}^{n} a_{ik}\, |A_{jk}| \tag{9}$$

$$= \delta_{ij} \det A \quad \leftarrow \text{using (7)}$$

Hence

$$(A \cdot \mathrm{Adj}\, A)_{ij} = \delta_{ij} \det A \tag{10}$$

Since this holds for all elements ij on both sides of the equation, we can write (10) as a matrix equation:

$$\underset{\text{matrix}}{A} \cdot \underset{\text{matrix}}{\mathrm{Adj}\, A} = \underset{\text{number}}{\det A} \cdot \underset{\text{matrix}}{I} \tag{11}$$

$$\Rightarrow \quad \cancel{A^{-1} A} \cdot \mathrm{Adj}\, A = A^{-1} \det A\, I \Rightarrow \boxed{A^{-1} = \frac{\mathrm{Adj}\, A}{\det A}} \tag{12}$$

As we have noted previously, $\mathrm{Adj}\, A$ is a matrix we can compute for any A. Hence the question of whether A^{-1} exists comes down to the question of whether $\det A = 0$.

CHANGE OF BASIS AND SIMILARITY

We have seen that a matrix (a_{ij}) represents an abstract linear transformation w.r.t. a specific basis. What then happens when we change basis? We can address this by asking the following questions:

Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ be two different bases for an n-dim. vector space V.

Q1: If $x = \sum_{i=1}^{n} \alpha_i x_i = \sum_{i=1}^{n} \beta_i y_i$ what is the relation between $\{\alpha_i\}$ and $\{\beta_i\}$ (1)

Q2: Given a set of constants $\{\alpha_i\}$ what is the relation between

$$x = \sum_{i=1}^{n} \alpha_i x_i \quad \text{and} \quad y = \sum_{i=1}^{n} \alpha_i y_i \tag{2}$$

ANSWERS: Since $\{x_i\}$ and $\{y_i\}$ are both bases there must be some linear transformation relating them:

$$\boxed{y_j = Ax_j = \sum_i a_{ij} x_i} \tag{3}$$

A1:

$$x = \sum_j \beta_j y_j = \sum_j \beta_j \left(\sum_i a_{ij} x_i \right) = \sum_i \left(\sum_j a_{ij} \beta_j \right) x_i \equiv \sum_i \alpha_i x_i \tag{4}$$

$$\boxed{\therefore \ \alpha_i = \sum_j a_{ij} \beta_j} \tag{5}$$

A2:

$$y = \sum_i \alpha_i y_i = \sum_i \alpha_i A x_i = A \sum_i \alpha_i x_i = Ax \tag{6}$$

$$\boxed{\therefore \ y = Ax} \tag{7}$$

Q3: Let L be a lin. transf. on V which has the following representations in the bases X and Y **F12***

$$X = \{x_i\} \qquad Y = \{y_i\}$$

$$L = \begin{cases} B = (b_{ij}) \text{ in } X \\ C = (c_{ij}) \text{ in } Y \end{cases} \tag{8}$$

an abstract lin. txfm.

$\Rightarrow$ What is the relationship between the matrices B and C?

A3:

$$B y_j = B(A x_j) = B\left(\sum_k a_{kj} x_k\right) = \sum_k a_{kj} B x_k \tag{9}$$

$$= \sum_k a_{kj} \underbrace{\left(\sum_i b_{ik} x_i\right)}_{\text{matrix rep. of } B \text{ w.r.t. } \{x_i\}} = \sum_i \left(\sum_k b_{ik} a_{kj}\right) x_i \tag{10}$$

However, the definition of (c_{ij}) is such that the linear transf. B when represented in terms of y_j is given by

$$B y_j \equiv \sum_k c_{kj} y_k = \sum_k c_{kj} A x_k = \sum_k c_{kj} \sum_i a_{ik} x_i \tag{11}$$

$\uparrow$ matrix rep. of A w.r.t. $\{x_i\}$

$$= \sum_i \left(\sum_k a_{ik} c_{kj}\right) x_i \tag{12}$$

Comparing (10) and (12) $\Rightarrow \sum_k b_{ik} a_{kj} = \sum_k a_{ik} c_{kj} \Rightarrow$ (13)

$$BA = AC$$

BA w.r.t. $\{x_i\}$, C w.r.t. $\{y_i\}$

$$\Rightarrow \boxed{C = A^{-1} B A} \tag{14}$$

C w.r.t. $\{y_i\}$, B w.r.t. $\{x_i\}$

This relates the matrix representation (C) of the abstract linear txfm. L given in the basis $\{y_i\}$ to its matrix rep. B given in terms of $\{x_i\}$ when the ~~basis~~ bases $\{x_i\}$ and $\{y_i\}$ are related to each other by $y_i = A x_i$ (so that A is also represented in the basis $\{x_i\}$). Thus C and B are representations of the same txfm w.r.t. different bases. Eq. (14) is a

$$\boxed{\text{SIMILARITY TRANSFORMATION}}$$

Q4: If (b_{ij}) is a matrix what is the relation between the linear transformations B and C defined by:

$$B x_j = \sum_i b_{ij} x_i \qquad C y_j = \sum_i b_{ij} y_i$$

A4:

$$\equiv C A x_j \Rightarrow$$

$$C y_j = \sum_i b_{ij} y_i = \sum_i b_{ij} (A x_i) = A \sum_i b_{ij} x_i = A B x_j$$

$\therefore \; C A x_j = A B x_j \Rightarrow \qquad CA = AB \Rightarrow \boxed{C = ABA^{-1}}$

Invariance of Matrices Under Similarity Transformations:

Let $C = A^{-1} B A$ then $\Rightarrow$

Thm:
a) $\det C = \det B$
b) $\operatorname{tr} C = \operatorname{tr} B$

Proof

a) $\det C = \det (A^{-1} B A) = (\det A^{-1})(\det B)(\det A)$

$= \underbrace{(\det A)(\det A^{-1})}_{\det(AA^{-1}) = \det I = 1} (\det B) = \det B$ ✓

b) $A = (a_{ij})$ $\qquad A^{-1} = (\bar{a}_{ij})$ $\qquad C = (c_{ij})$ $\quad B = (b_{ij})$

$AA^{-1} = I \Rightarrow \sum_k a_{ik} \bar{a}_{kj} = \delta_{ij}$

$C = A^{-1} B A \Rightarrow c_{ij} = \sum_{l,k} \bar{a}_{il} b_{lk} a_{kj} \Rightarrow \operatorname{Tr} C = \sum_i c_{ii} = \sum_{i,l,k} \bar{a}_{il} b_{lk} a_{ki}$

$= \sum_{l,k} \underbrace{\left(\sum_i a_{ki} \bar{a}_{il}\right)}_{\delta_{kl}} b_{lk} = \sum_l b_{ll} = \operatorname{Tr} B$ ✓

THE EIGENVALUE PROBLEM

In many physical systems, particularly QM we confront the equation

$$\text{Operator} \rightarrow \underset{\uparrow\ \text{eigenvector}}{M x} = \underset{\uparrow\ \text{eigenvalue}}{\lambda} x \equiv \lambda I x \qquad (1)$$

The eigenvalue problem involves finding the non-trivial solutions to (1).

$$(1) \Rightarrow \quad \underbrace{(M - \lambda I)}_{A} x = 0 = Ax \qquad (2)$$

As noted previously (2) has non-trivial solutions only when A^{-1} does not exist $\Rightarrow$ $\det A = 0$

$$\boxed{\therefore \text{ non-trivial} \Rightarrow \det(M - \lambda I) = 0} \quad (3)$$

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix} \Rightarrow \underbrace{\det(M - \lambda I) = \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ a_{31} & & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} - \lambda \end{pmatrix} = 0}_{\text{Characteristic equation}} \quad (4)$$

polynomial of order λ^n $\Rightarrow$ n roots (not ~~necessarily~~ necessarily distinct)

I $\boxed{(\lambda_1 \cdots \lambda_n) = \text{Spectrum of } M}$ (5)

II Given $\{\lambda_n\}$, only certain eigenvectors x will solve the original equation (1). These are found by

$$M x_1 = \lambda_1 x_1 \qquad M x_2 = \lambda_2 x_2 \quad \cdots \qquad (6)$$

Superposition (key in QM)

$$M x_1 = \lambda_1 x_1 \qquad M x_2 = \lambda_1 x_2 \Rightarrow M(\alpha x_1 + \beta x_2) = \alpha M x_1 + \beta M x_2 \qquad (7)$$

$$= \alpha \lambda_1 x_1 + \beta \lambda_1 x_2 = \lambda_1 (\alpha x_1 + \beta x_2) \Rightarrow \underline{\alpha x_1 + \beta x_2 = \text{eigenvector too}}$$