INFINITE SERIES

Why do we care?

(a) Many familiar functions are represented as an infinite series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \qquad (1)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \; ; \; \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (2)$$

$$\hookrightarrow x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \hookrightarrow = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad (3)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad (4)$$

(b) The FROBENIUS METHOD for solving differential equations naturally leads to an infinite series, and so we must be careful when using it to consider the question of convergence of the series.

Since this method has wide applications, we introduce it here to help motivate the ensuing discussion of series.

Consider the differential equation for the SIMPLE HARMONIC OSCILLATOR: The amplitude $y(x)$ is given by [we use x here instead of t]

$$y''(x) + \omega^2 y(x) = 0, \qquad \omega = \text{constant} \qquad (5)$$

We know that the solution(s) to (5) will be $\sin(\omega x)$ and $\cos(\omega x)$

So let us see how this comes about.

We begin by assuming that ($k \equiv$ index)

$$y(x) = x^k (a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} \qquad (6)$$

where we define $a_0 \neq 0$ (a_0 is the coefficient of the first nonzero term).

We will eventually determine k and all the constants a_λ. Then (6) $\Rightarrow$

$$y'(x) = \sum_{\lambda=0}^{\infty} (k+\lambda) a_\lambda x^{k+\lambda-1} \;;\quad y''(x) = \sum_{\lambda=0}^{\infty} (k+\lambda)(k+\lambda-1) a_\lambda x^{k+\lambda-2} \qquad (7)$$

Combining (5)–(7): $$\sum_\lambda (k+\lambda)(k+\lambda-1) a_\lambda x^{k+\lambda-2} + \omega^2 \sum_\lambda a_\lambda x^{k+\lambda} = 0 \qquad (8)$$

In order for this equation to hold each term in the series (i.e. the coefficients of each power of x) must separately vanish. To see how this comes about, write out the first few terms:

$$\underbrace{\left[k(k-1)a_0 x^{k-2} + \omega^2 a_0 x^k\right]}_{\lambda=0} + \underbrace{\left[(k+1)k a_1 x^{k-1} + \omega^2 a_1 x^{k+1}\right]}_{\lambda=1} \qquad (9)$$

$$+ \underbrace{\left[(k+2)(k+1)a_2 x^k + \omega^2 a_2 x^{k+2}\right]}_{\lambda=2} + \underbrace{\left[(k+3)(k+2)a_3 x^{k+1} + \omega^2 a_3 x^{k+3}\right]}_{\lambda=3} + \dots$$

We see from (9) that the lowest power of x, which is x^{k-2} appears in only one term. Since this term must vanish by itself even though $a_0 \neq 0$ (by definition) it follows that k must satisfy

$$\boxed{k(k-1) = 0 \Rightarrow k = 0, 1} \qquad (10)$$

INDICIAL EQUATION

The indicial equation always arises in this way from the lowest term.

We next demand that the coefficients of the remaining independent powers of x also vanish. We note from (9) that the term $\underline{\lambda = j}$ in [the second term of] Eq. (8) will give rise to the same power of x as the term $\underline{\lambda = j+2}$ in the first term of Eq. (8), in both cases this being x^{k+j}.

Thus the net coefficient of x^{k+j} is given by

$$[(k+j+2)(k+j+2-1)\,a_{j+2} + \omega^2 a_j]\,x^{k+j} \equiv 0 \Rightarrow \qquad (11)$$

$$\boxed{a_{j+2} = \frac{-\omega^2 a_j}{(k+j+2)(k+j+1)}} \qquad (12)$$

<u>RECURRENCE RELATION</u>

Together, the indicial equation ⊕ recurrence relation determine one solution of the original differential equation. From (12) it follows that a_2 is given in terms of a_0, a_4 in terms of a_2, ..., where <u>a_0 itself is not determined</u> (it will be one of 2 unknown constants that arise for a 2ND order differential equation). This solution has only even powers of x, and we are free at this point to set $a_1 \equiv 0$ (Why? <u>Because this works!</u> ---. See below!!) Combining the indicial equation (10) & the recurrence relation (12), we <u>start with $k=0$</u>)

Then (12) $\Rightarrow$ $a_{j+2} = \dfrac{-\omega^2 a_j}{(j+2)(j+1)} \Rightarrow a_2 = \dfrac{-\omega^2 a_0}{2\cdot 1} = \dfrac{-\omega^2 a_0}{2!}$ (13)

$$a_4 = \frac{-\omega^2 a_2}{4\cdot 3} = +\frac{\omega^4 a_0}{4!}\;;\quad a_6 = \frac{-\omega^2 a_4}{6\cdot 5} = \frac{-\omega^6 a_0}{6!}, \;\ldots \text{ etc.} \qquad (14)$$

$$\therefore\ \boxed{a_{2n} = (-1)^n \frac{\omega^{2n} a_0}{(2n)!}} \qquad (15)$$

From (15) with $k=0$:

$$y(x) = x^0 a_0 \left\{1 - \frac{1}{2!}\omega^2 x^2 + \frac{1}{4!}\omega^4 x^4 - \frac{1}{6!}\omega^6 x^6 + \dots\right\}$$

$$\therefore \boxed{y(x) = a_0 \cos \omega x} \longleftrightarrow \boxed{k=0} \qquad (16)$$

Next we examine the solution corresponding to $k=1$ in the indicial equation:
From (12) [the recurrence relation] $k=1 \Rightarrow$

$$a_{j+2} = \frac{-\omega^2 a_j}{(j+3)(j+2)} \Rightarrow a_2 = \frac{-\omega^2 a_0}{3 \cdot 2}\ ;\ a_4 = \frac{-\omega^2 a_2}{5 \cdot 4} = \frac{+\omega^4 a_0}{5!}$$

$$a_6 = \frac{-\omega^2 a_4}{7 \cdot 6} = -\frac{\omega^6 a_0}{7!}\ ;\ \dots \text{ etc.} \qquad (17)$$

For the $k=1$ solution then: $$\boxed{a_{2n} = \frac{(-1)^n \omega^{2n} a_0}{(2n+1)!}} \qquad (18)$$

The solution for $k=1$ is then $y = x^1(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots)$ (19)

$$\therefore y(x) = a_0 x \left\{1 - \frac{\omega^2 x^2}{3!} + \frac{\omega^4 x^4}{5!} - \frac{\omega^6 x^6}{7!} + \dots\right\} \qquad (20)$$

$$y = a_0 \left\{x - \frac{\omega^2 x^3}{3!} + \frac{\omega^4 x^5}{5!} - \frac{\omega^6 x^7}{7!} + \dots\right\} \qquad (21)$$

$$\boxed{\therefore y(x) = \frac{a_0}{\omega}\left\{\omega x - \frac{\omega^3 x^3}{3!} + \frac{\omega^5 x^5}{5!} - \frac{\omega^7 x^7}{7!} + \dots\right\} = \frac{a_0}{\omega} \sin \omega x} \qquad (22)$$

Thus in this situation $k=1$ gives a second solution in terms of another undetermined constant [This a_0 is not necessarily the same as previously.]

COMMENTS ON SOLUTIONS:

[1] We will later formally prove that $\sin \omega x$ and $\cos \omega x$ are independent solutions. Knowing this we conclude that in this case (but not always!) the 2 solutions of the indicial equation gave us 2 linearly independent solutions. Since there can be at most 2 lin. indep. solutions, we have solved the equation completely. Thus we did not "lose" anything by the assumption $a_1 \equiv 0$.

[2] In general we should verify that the solutions actually solve the original equation, although here this is trivial.

[3] When a series solution leads to an infinite series, we should formally test for convergence.

DEFINITION(S) OF CONVERGENCE

Consider a sum: $S_N \equiv \sum_{n=0}^{N} a_n$ (1)

(In some examples the sum may start with $n=1$, or another integer)

If n is finite then S_N will also be finite, provided there are no singularities in the n-th term a_n, which could be some expression like $a_n = 1/n^2$ or, as in the case of e^x, $a_n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

The series in (1) is said to converge (uniformly) to S if

$$\boxed{|S - S_N| < \epsilon \quad \text{for } N > N(\epsilon)} \qquad (2)$$

In simple English, if the sum S_N in (1) can be made to approximate S within a tolerance $\epsilon \ll 1$, when $N > N(\epsilon)$, then the series is said to converge uniformly to S. The idea is that however small ϵ is chosen to be, there must be some $N = N(\epsilon)$ for which (2) holds if S_N converges to S.

As formulated in (2) the convergence criterion assumes that we know what the exact sum S is. But much of the time we do not really care what S is, or even know what S is; all we care about is whether or not S_N approaches a finite value as $N \to \infty$. It turns out that we can replace (2) by the CAUCHY CRITERION:

integer

$$\boxed{|S_N - S_M| < \epsilon \quad \text{for } N, M > I(\epsilon)} \qquad (3)$$

Let us begin by examining an infinite series which does converge. Consider

$$S = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots=\sum_{n=0}^{\infty}\frac{1}{2^n} \qquad (4)$$

This is an example of a geometric series of the form

$$S_N = 1+r^2+r^3+\dots+r^N \qquad (5)$$

where each term is smaller than the preceding term by a factor of r. This series can be summed exactly by the following procedure: (5)⇒

$$rS_N = r+r^3+r^4+\dots+r^{N+1} \Rightarrow S_N - rS_N = 1-r^{N+1} \qquad (6)$$

$$\Rightarrow S_N(1-r) = 1-r^{N+1} \Rightarrow \boxed{S_N = \frac{1-r^{N+1}}{1-r} = \frac{1}{1-r} - \frac{r^{N+1}}{1-r}} \qquad (7)$$

Eq. (7) is an exact result for all N. For finite N clearly the result S_N must also be finite, so the question is what happens as $N\to\infty$. We see from (7) that as $N\to\infty$ $r^{N+1}\to 0$ since $|r|<1$. In this limit We then find

$$S \equiv S_{N\to\infty} = \frac{1}{1-r} \xrightarrow[r=1/2]{} = 2 \qquad (8)$$

This can be understood intuitively by returning to (4) and writing

$$S-1 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots = 1 \qquad (9)$$

Suppose that you start at a certain distance from a wall, say 1 m.

What (9) says is that if you walk half way to the wall on each successive step, then after an infinite number of steps you will have covered exactly 1 m, and reached the wall.

(This is the response to one of ZENO'S PARADOXES)

We now want to connect this result to the definition of convergence that we have given earlier, the condition

$$|S_\infty - S_N| < \epsilon \quad \text{when } N > N(\epsilon) \qquad (10)$$

From (7) & (8) $$|S_\infty - S_N| = \left|\left(\cancel{\frac{1}{1-r}}\right) - \left(\cancel{\frac{1}{1-r}} - \frac{r^{N+1}}{1-r}\right)\right| = \left|\frac{r^{N+1}}{1-r}\right| \qquad (11)$$

From (10) & (11) we want to find the value of $N \equiv N(\epsilon)$ such that

$$\left|\frac{r^{N+1}}{1-r}\right| = \frac{r^{N+1}}{1-r} = \left(\frac{e^{(N+1)\ln r}}{1-r}\right) < \epsilon \qquad (12)$$

To solve for N: $\ln(\quad) < \ln\epsilon \Rightarrow (N+1)\ln r - \ln(1-r) < \ln\epsilon$

$$\Rightarrow (N+1)\ln r = \ln\epsilon + \ln(1-r) = \ln\{\epsilon(1-r)\} \qquad (13)$$

$$\therefore (N+1) = \frac{\ln\{\epsilon(1-r)\}}{\ln r} \Rightarrow \boxed{N = \frac{\ln\{\epsilon(1-r)\}}{\ln r} - 1} \qquad (14)$$

We note that if $r<1 \ni \ln\{\epsilon(1-r)\}$ is finite and (14) can always be realized, and this $\Rightarrow$ that the original series is convergent; all we have to do is to make $N = N(\epsilon)$ in (10).

TECHNIQUES FOR SUMMING SOME SERIES

Consider the class of series given by

$$S_p(N) = \sum_{n=1}^{N} n^p \quad ; \; p = \text{integer} \qquad (1)$$

This class of series can be summed by the following recurrence method:

Begin with $p=0$:

$$\boxed{S_0(N) = \sum_{n=1}^{N} n^0 = \sum_{n=1}^{N} 1 = N} \qquad \text{(obvious!)} \qquad (2)$$

To evaluate $S_1(N) = \sum_{n=1}^{N} n$ consider the following expression:

$$\sum_{n=1}^{N} [(n+1)^2 - n^2] = [\cancel{2^2} - 1^2] + [\cancel{3^2} - \cancel{2^2}] + [4^2 - \cancel{3^2}] + \ldots + \qquad (3)$$

$$+ [\cancel{(N-1+1)^2} - (N-1)^2] + [(N+1)^2 - \cancel{N^2}] = -1 + (N+1)^2$$

The key point in Eq. (3) is that the only terms which do <u>not</u> get cancelled are the very first and the very last. Hence

$$\sum_{n=1}^{N} [(n+1)^2 - n^2] = -1 + (N+1)^2 = \cancel{-1} + (N^2 + 2N + \cancel{1}) = \underline{\underline{N^2 + 2N}} \qquad (4)$$

$$\Downarrow$$

$$\sum_{n=1}^{N} [\cancel{n^2} + 2n + 1 - \cancel{n^2}] = \sum_{n=1}^{N} [2n+1] = \underbrace{2\sum_{n=1}^{N} n}_{2S_1(N)} + \underbrace{\sum_{n=1}^{N} 1}_{N} \qquad (5)$$

Combining (4) & (5) ⇒

$$2S_1(N) + N = N^2 + 2N \qquad (6)$$

$$\Rightarrow \quad \boxed{S_1(N) = \sum_{n=1}^{N} n = \frac{1}{2} N(N+1)} \qquad (7)$$

As a check on $S_1(N)$:

$$S_1(4) = 1+2+3+4 = 10$$

$$\stackrel{?}{=} \frac{1}{2}\cdot 4 \cdot 5 = 10 \checkmark \qquad (8)$$

$$S_1(5) = 15 \stackrel{?}{=} \frac{1}{2}\cdot 5 \cdot 6 = 15 \checkmark \qquad (9)$$

This recurrence method can be extended to evaluate higher-order sums:

$$S_2(N) = \sum_{n=1}^{N} n^2 \Rightarrow \text{Consider } \sum_{n=1}^{N}\left[(n+1)^3 - n^3\right] = [2^3 - 1^3] + \dots + [(N+1)^3 - N^3]$$

$$= -1^3 + (N+1)^3 = \cancel{-1} + (N^3 + 3N^2 + 3N \cancel{+1}) \qquad (10)$$

lowest highest

$$\therefore \sum_{n=1}^{N}\left[(n+1)^3 - n^3\right] = \underline{\underline{N^3 + 3N^2 + 3N}} \qquad (11)$$

$$\sum_{n=1}^{N}\left[\cancel{n^3} + 3n^2 + 3n + 1 - \cancel{n^3}\right] = 3\sum_{n=1}^{N} n^2 + 3\sum_{n=1}^{N} n + \sum_{n=1}^{N} 1 \qquad (12)$$

$$= 3S_2(N) + 3S_1(N) + S_0(N) = \underline{\underline{N^3 + 3N^2 + 3N}}$$

Combining (11) & (12) with the previous results gives:

$$3S_2(N) + 3\cdot\frac{1}{2}N(N+1) + N = N^3 + 3N^2 + 3N \qquad (13)$$

Solving for $S_2(N)$ gives:

$$\boxed{S_2(N) = \sum_{n=1}^{N} n^2 = \frac{N(2N^2+3N+1)}{6} = \frac{1}{6}N(N+1)(2N+1)} \qquad (14)$$

In a similar way we can show that

$$\boxed{S_3(N) = \sum_{n=1}^{N} n^3 = \frac{1}{4}N^2(N+1)^2 = \left[\frac{N(N+1)}{2}\right]^2 \qquad (15)}$$

TESTS FOR CONVERGENCE

We begin with numerical series (no variables involved!) and review various tests for convergence.

Consider the generic series: $S = \sum_{n=1}^{\infty} a_n$ if this is finite, there is no issue! (1)

Comparison test:

Suppose there is another series $S' = \sum_{n=1}^{\infty} b_n$ (2)

Then if $a_n \le b_n$ & S' converges, then so does S. Note that it is possible for $a_n > b_n$ for any finite number of terms, and still have a_n converge; what is critical is the pattern of terms as $n \to \infty$.

As noted in the text, even though $a_n = 1000/n^2 > b_n = 1/1000n$ for some finite number of terms, the a_n series converges, whereas the b_n series does not!!

APPLICATION: The Riemann Zeta function is defined by

$$\boxed{\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad s > 1} \tag{3}$$

It can be shown that $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. For our purposes we do not wish to evaluate $\zeta(2)$ exactly, but merely to show that $\zeta(2)$ converges to a finite number. We can do this by grouping the terms in the series in such a way that the first term in each grouping is part of a geometric series:

NUMBER OF TERMS: → 2 4 8 + ...

$$\zeta(2) = 1 + \left(\tfrac{1}{4} + \tfrac{1}{9}\right) + \left(\tfrac{1}{16} + \tfrac{1}{25} + \tfrac{1}{36} + \tfrac{1}{49}\right) + \left(\tfrac{1}{64} + \dots \tfrac{1}{225}\right) + \dots \quad (4)$$

$$\tfrac{1}{1^2} + \left(\tfrac{1}{2^2} + \tfrac{1}{3^2}\right) + \left(\tfrac{1}{4^2} + \tfrac{1}{5^2} + \tfrac{1}{6^2} + \tfrac{1}{7^2}\right) + \left(\tfrac{1}{8^2} + \tfrac{1}{9^2} + \dots \tfrac{1}{15^2}\right) + \dots$$

Note that in each term shown in RED there is a power of $\frac{1}{4}$.

Note also that in each (...) this term is the largest. Hence if we can show that the series containing only the red terms converges, then $\zeta(2)$ itself must converge. From the previous work we have shown that

$$S_N = 1 + r + r^2 + \dots + r^N = \frac{1}{1-r} - \frac{r^{N+1}}{1-r} \xrightarrow{N \to \infty} \frac{1}{1-r} \quad (5)$$

So the terms in RED would converge to $S_N = \dfrac{1}{1-\frac{1}{4}} = \dfrac{4}{3}$ (6)

But this is not the end of the story!! We have also to take account of how many terms there are in each (...) being replaced by $\frac{1}{4}$, $\frac{1}{16}$, etc. You will do this for homework.

APPLICATION 2:

For homework you will show that the harmonic series

$$S = \sum_{n=1}^{\infty} \frac{1}{n} \longrightarrow \infty$$

APPLICATION 3: text problem 4.2.3

Q: Given that $\sum a_n = \sum 1/n$ diverges, what can you say about $\sum 1/1000\sqrt{n}$?

A: It diverges, since for $\sqrt{n} > 1000$ $(n > 10^6)$ each term in the harmonic series is smaller than in this series, and yet the harmonic series still diverges.

THE RATIO TEST:

change in notation from text!!

This test asserts that if $\boxed{R \equiv \lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1}$ (1)

Then the series converges.

Note that for the previously considered geometric series

$$S = 1 + r + r^2 + \dots \Rightarrow \frac{a_{n+1}}{a_n} = \frac{r^{n+1}}{r^n} = r \qquad (2)$$

Hence if $r<1$ then $R = r < 1$, and the series converges.

However, by contrast, for the harmonic series $\sum_n a_n = \sum_n \frac{1}{n}$ (3)

We find

$$R = \lim_{n\to\infty} \frac{1/n+1}{1/n} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \qquad (4)$$

Since R is not strictly less than 1, we cannot conclude from the ratio test alone that the harmonic series converges. We have in fact already shown - see homework - that it does not.

Intuitively this makes sense since successive terms look more and more like 1, So we are adding $1+1+1+1 \dots$

THE INTEGRAL TEST:

We originally developed our understanding of integrals by viewing them as sums of rectangles of width Δx, in the limit as $\Delta x \to 0$. The deep connection between sums and integrals allows us to go back and forth between them.

The test: $\sum_{n=1}^{\infty} f(n)$ can be evaluated for convergence by considering $I = \int_1^L f(x)\,dx$ as $L \to \infty$. If I converges so does the $\sum_n$; likewise, if I diverges, so does $\sum_n$. This can be understood by considering the figure in Fig. 4.1 of the text.

Applying the integral test to the harmonic series we find

$$\lim_{L\to\infty} \int_1^L \frac{1}{x}\,dx = \ln x\Big|_1^L = \ln L - \underbrace{\ln 1}_{0} = \ln L \to \infty \qquad (1)$$

Hence, by virtue of the integral test we can conclude that the harmonic series diverges.

By way of contrast $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since

$$\lim_{L\to\infty} \int_1^L \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{x=1}^{x=L} = 1 - \frac{1}{L} \longrightarrow 1 = \text{finite}$$

The integral test is very useful in practice because it is usually easier to evaluate integrals than it is to sum discrete sums.

SOME SHORT CUTS:

As noted previously, the only functional dependence that we care about is what happens as $n \to \infty$. Whatever happens for finite n is not a concern. Consider then a function $f(n)$ of the form

$$f(n) = \frac{g(n)}{h(n)} \xrightarrow{n \to \infty} \frac{\text{biggest terms in } g(n)}{\text{biggest term in } h(n)} \qquad (1)$$

Using the example in the text:

$$f(n) = \frac{3n^2 + 4n + 6}{n^4 + 12n^3 + 4n + 2} \longrightarrow \frac{3n^2}{n^4} \to \frac{3}{n^2} \text{ as } n \to \infty. \quad (2)$$

Since we have shown that $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = \text{finite}$ (3)

it follows that the expression in (2) will also be finite as we evaluate $\sum f(n)$.

ABSOLUTE VERSUS CONDITIONAL CONVERGENCE

This relates to series which have terms which alternate in sign.

Definition: A series containing both positive and negative terms is said to converge absolutely if the sum with all terms replaced by their absolute values also converges.

Comment: This makes sense intuitively, since the series with both positive and negative terms is certainly going to sum to a smaller value due to cancellations.

Example: Consider the series: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ (1)

The above discussion indicates that this series will converge if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. But this is, again, just $\zeta(2) = \pi^2/6$, and hence the series in (1) also converges. In fact it converges to $\pi^2/12$ which is indeed smaller than $\zeta(2) = \pi^2/6$.

Definition: A series which converges when the actual signs are included, but not when they are replaced by absolute values is said to be <u>conditionally convergent</u>.