Behavior of the Affine Connection Under Coordinate Transformations

We have noted that the significance of $\Gamma^{\sigma}_{\lambda\mu}$ in part stems from the fact that it is not a tensor, which means that it does not transform properly under a change of coords. We now show this: Start with

$$\Gamma'^{\lambda}_{\mu\nu}(x') \equiv \left(\frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}}\right)\left[\frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu}\partial x'^{\nu}}\right] = \left(\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\right)\left[\frac{\partial}{\partial x'^{\mu}}\cdot\frac{\partial \xi^{\alpha}}{\partial x'^{\nu}}\right] \tag{12}$$

We want to end up with terms like this which eventually give Γ in the x cood system

$$\Gamma'^{\lambda}_{\mu\nu}(x') = \left(\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\right)\left[\frac{\partial}{\partial x'^{\mu}}\left(\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right)\right] \tag{13}$$

$$= (\quad)\left[\frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu}\partial x^{\sigma}}\left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right) + \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\cdot\frac{\partial^2 x^{\sigma}}{\partial x'^{\mu}\partial x'^{\nu}}\right] \tag{14}$$

$$\to \frac{\partial^2 \xi^{\alpha}}{\partial x^{\sigma}\partial x^{\tau}}\cdot\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \tag{15}$$

$$\to \frac{\partial \xi^{\alpha}}{\partial x^{\epsilon}}\Gamma^{\epsilon}_{\sigma\tau}$$

Collecting everything together gives

$$\Gamma'^{\lambda}_{\mu\nu}(x') = \left(\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\right)\left[\frac{\partial \xi^{\alpha}}{\partial x^{\epsilon}}\Gamma^{\epsilon}_{\sigma\tau}\,\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} + \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\frac{\partial^2 x^{\sigma}}{\partial x'^{\mu}\partial x'^{\nu}}\right] \quad (16)$$

$$\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\left(\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\frac{\partial \xi^{\alpha}}{\partial x^{\epsilon}}\right)\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Gamma^{\epsilon}_{\sigma\tau} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Gamma^{\rho}_{\sigma\tau} \quad (17)$$

$$\frac{\partial x^{\rho}}{\partial x^{\epsilon}} = \delta^{\rho}_{\epsilon}$$

Hence $$\Gamma'^{\lambda}_{\mu\nu}(x') = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\Gamma^{\rho}_{\sigma\tau} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\left(\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\right)\frac{\partial^2 x^{\sigma}}{\partial x'^{\mu}\partial x'^{\nu}} \quad (18)$$

$$\frac{\partial x^{\rho}}{\partial x^{\sigma}} = \delta^{\rho}_{\sigma}$$

Finally!! $$\boxed{\Gamma'^{\lambda}_{\mu\nu} = \left(\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\right)\left(\frac{\partial x^{\sigma}}{\partial x'^{\mu}}\right)\left(\frac{\partial x^{\tau}}{\partial x'^{\mu}}\right)\Gamma^{\rho}_{\sigma\tau} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}}} \quad (19)$$

"expected" "extra piece"

The presence of the "extra piece" demonstrates why $\Gamma'^{\lambda}_{\mu\nu}$ is not a tensor.

<u>A Useful Identity</u>: Start with $\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial x'^{\mu}} = \delta^{\lambda}_{\mu}$ $(\mu \to \nu)$ (20)

and differentiate w.r.t. x'^{μ}:

$$\frac{\partial^2 x'^{\lambda}}{\partial x'^{\mu}\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial x'^{\nu}} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}} = \frac{\partial}{\partial x'^{\mu}}\left(\delta^{\lambda}_{\nu}\right) = 0 \quad (21)$$

$$\frac{\partial^2 x'^{\lambda}}{\partial x^{\rho}\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x^{\rho}}{\partial x'^{\nu}}$$

"extra piece" in (19)

It follows that we can replace the "extra piece" in C(19) by the first term in (21) giving (note (−) sign)

$$\boxed{\Gamma'^{\lambda}_{\mu\nu}(x') = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\Gamma^{\rho}_{\sigma\tau} - \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho}\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x^{\rho}}{\partial x'^{\nu}}} \qquad (22)$$

COVARIANT DIFFERENTIATION

To show the need for covariant differentiation we show that conventional partial derivatives do not produce tensors:

$$V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}(x) \qquad (23)$$

$$\frac{\partial}{\partial x'^{\lambda}} V'^{\mu}(x') = \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\nu}}\left(\frac{\partial V^{\nu}(x)}{\partial x'^{\lambda}}\right)}_{\text{change in vector}} + \underbrace{\frac{\partial^2 x'^{\mu}}{\partial x^{\nu}\partial x'^{\lambda}} V^{\nu}(x)}_{\text{change in coord system}} \qquad (24)$$

$$\boxed{\therefore \frac{\partial}{\partial x'^{\lambda}} V'^{\mu}(x') = \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\nu}}\frac{\partial x^{\rho}}{\partial x'^{\lambda}}\left(\frac{\partial V^{\nu}(x)}{\partial x^{\rho}}\right)}_{\text{"expected"}} + \underbrace{\frac{\partial^2 x'^{\mu}}{\partial x^{\nu}\partial x^{\rho}}\left(\frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}(x)\right)}_{\text{"extra piece"}}} \qquad (25)$$

Constructing the Covariant Derivative

Having shown that $\frac{\partial}{\partial x'^\lambda} V'^\mu(x')$ and $\Gamma'^\lambda_{\mu\nu}(x')$ both have "extrapieces" left over that prevent each from behaving as proper tensors, we seek to combine these to make these extra pieces "go away". Consider:

$$\left(\Gamma'^\mu_{\lambda\kappa}\right)\left[V'^\kappa\right] = \left(\frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\kappa}\frac{\partial x^\sigma}{\partial x'^\lambda}\Gamma^\nu_{\rho\sigma} - \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\nu}{\partial x'^\kappa}\right)\left[\frac{\partial x'^\kappa}{\partial x^\eta} V^\eta\right] \quad (5)$$

$$= \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\kappa}\frac{\partial x^\sigma}{\partial x'^\lambda}\Gamma^\nu_{\rho\sigma}\frac{\partial x'^\kappa}{\partial x^\eta}V^\eta - \frac{\partial^2 x'^\mu}{\partial x^\nu\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x^\nu}{\partial x'^\kappa}\frac{\partial x'^\kappa}{\partial x^\eta}V^\eta \quad (6)$$

$$\frac{\partial x^\rho}{\partial x^\eta} = \delta^\rho_\eta \qquad \frac{\partial x^\nu}{\partial x^\eta} = \delta^\nu_\eta$$

$$\boxed{\text{Hence: } \Gamma'^\mu_{\lambda\kappa}(x')V'^\kappa(x') = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\sigma}{\partial x'^\lambda}\Gamma^\nu_{\rho\sigma}(x)V^\rho(x) - \frac{\partial^2 x'^\mu}{\partial x^\nu\partial x^\rho}\left(\frac{\partial x^\rho}{\partial x'^\lambda}V^\nu(x)\right)} \quad (7)$$

If we combine this result with Eq. (25) p. 59 we find

$$\frac{\partial}{\partial x'^\lambda}V'^\mu(x') + \Gamma'^\mu_{\lambda\kappa}(x')V'^\kappa(x') = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\rho}{\partial x'^\lambda}\left(\frac{\partial V^\nu(x)}{\partial x^\rho}\right) + \cancel{\frac{\partial^2 x'^\mu}{\partial x^\nu\partial x^\rho}\left(\frac{\partial x^\rho}{\partial x'^\lambda}V^\nu(x)\right)}$$

$$+ \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\sigma}{\partial x'^\lambda}\left(\Gamma^\nu_{\rho\sigma}(x)V^\rho(x)\right) - \cancel{\frac{\partial^2 x'^\mu}{\partial x^\nu\partial x^\rho}\left(\frac{\partial x^\rho}{\partial x'^\lambda}V^\nu(x)\right)}$$

NOTE! The extrapieces have cancelled!!

Collecting together the previous results:

$$\boxed{\left(\frac{\partial}{\partial x'^{\lambda}} V'^{\mu}(x') + \Gamma'^{\mu}_{\lambda\kappa}(x') V'^{\kappa}(x')\right) = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \left(\frac{\partial}{\partial x^{\rho}} V^{\nu}(x) + \Gamma^{\nu}_{\rho\sigma}(x) V^{\sigma}(x)\right)}$$

2ND RANK TENSOR x'

CORRECT TRANFORMATION MATRICES

2ND RANK TENSOR IN x

The expressions in ( ) are defined as the <u>COVARIANT DERIVATIVE OF THE CONTRAVARIANT VECTOR V'^{μ}</u>

<u>TERMINOLOGY:</u> "COVARIANT" means 2 things in tensor analysis:

a) Refers to a vector such as $U_{\mu} = \partial\phi/\partial x^{\mu}$

b) Refers to a quantity which transforms properly when going from one coordinate system to another

COVARIANT DERIVATIVE OF A COVARIANT VECTOR $U_{\mu}(x)$:

$$\boxed{\left(\frac{\partial U'_{\mu}(x')}{\partial x'^{\lambda}} - \Gamma'^{\nu}_{\mu\lambda}(x') U'_{\nu}(x')\right) = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\lambda}} \left(\frac{\partial U_{\rho}(x)}{\partial x^{\sigma}} - \Gamma^{\tau}_{\sigma\rho}(x) U_{\tau}(x)\right)}$$

<u>NOTATION:</u> $\left(\frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}_{\lambda\kappa} V^{\kappa}\right) \equiv D_{\lambda} V^{\mu} \equiv V^{\mu}_{;\lambda} = V^{\mu}_{\|\lambda}$; $\frac{\partial V^{\mu}}{\partial x^{\lambda}} \equiv V^{\mu}_{,\lambda} \equiv V^{\mu}_{|\lambda} \equiv \partial_{\lambda} V^{\mu}$

$\left(\frac{\partial U_{\mu}}{\partial x^{\lambda}} - \Gamma^{\kappa}_{\mu\lambda} U_{\kappa}\right) \equiv D_{\lambda} U_{\mu} \equiv U_{\mu;\lambda} = U_{\mu\|\lambda}$; $\frac{\partial U_{\mu}}{\partial x^{\lambda}} \equiv U_{\mu,\lambda} = U_{\mu|\lambda} \equiv \partial_{\lambda} U_{\mu}$

The latter notation helps as a mnemonic for the indices.

COVARIANT DERIVATIVE OF AN ARBITRARY TENSOR

$$D_\rho T^{\mu\sigma}_{\lambda} \sim D_\rho (V^\mu W^\sigma U_\lambda) = (D_\rho V^\mu) W^\sigma U_\lambda + V^\mu (D_\rho W^\sigma) U_\lambda + V^\mu W^\sigma (D_\rho U_\lambda) \tag{18}$$

$$= (\partial_\rho V^\mu + \Gamma^\mu_{\rho\kappa} V^\kappa) W^\sigma U_\lambda + (\partial_\rho W^\sigma + \Gamma^\sigma_{\rho\kappa} W^\kappa) V^\mu U_\lambda + V^\mu W^\sigma (\partial_\rho U_\lambda - \Gamma^\kappa_{\rho\lambda}) U_\kappa \tag{19}$$

$$= \underbrace{\left\{ \partial_\rho V^\mu \cdot W^\sigma U_\lambda + \partial_\rho W^\sigma \cdot V^\mu U_\lambda + \partial_\rho U_\lambda V^\mu W^\sigma \right\}}_{} \rightarrow = \partial_\rho (V^\mu W^\sigma U_\lambda) = \partial_\rho T^{\mu\sigma}_{\lambda} \equiv T^{\mu\sigma}_{\lambda,\rho}$$

$$+ \Gamma^\mu_{\rho\kappa} \underbrace{V^\kappa W^\sigma U_\lambda}_{T^{\kappa\sigma}_{\lambda}} + \Gamma^\sigma_{\rho\kappa} \underbrace{V^\mu W^\kappa U_\lambda}_{T^{\mu\kappa}_{\lambda}} - \Gamma^\kappa_{\rho\lambda} \underbrace{V^\mu W^\sigma U_\kappa}_{T^{\mu\sigma}_{\kappa}} \tag{20}$$

$$\boxed{\text{Hence: } D_\rho T^{\mu\sigma}_{\lambda} \equiv T^{\mu\sigma}_{\lambda;\rho} = T^{\mu\sigma}_{\lambda,\rho} + \Gamma^\mu_{\rho\kappa} T^{\kappa\sigma}_{\lambda} + \Gamma^\sigma_{\rho\kappa} T^{\mu\kappa}_{\lambda} - \Gamma^\kappa_{\rho\lambda} T^{\mu\sigma}_{\kappa}} \tag{21}$$

An Application: (The "real" Metric Compatibility Condition")

Consider $g_{\mu\nu;\lambda} \equiv D_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa_{\nu\lambda} g_{\mu\kappa} = 0^*$

* p. 55 Eq. (16)

Alternatively: $g'_{\mu\nu;\lambda} = \dfrac{\partial \xi^\alpha}{\partial x'^\mu} \dfrac{\partial \xi^\beta}{\partial x'^\nu} \dfrac{\partial x^\gamma}{\partial x'^\lambda} \underbrace{\eta_{\alpha\beta;\gamma}}_{\equiv\, 0} = 0$

Covariant Derivatives do NOT Commute!

This is part of the reason why they are so interesting!

Consider $(D_\alpha D_\beta)V^\mu - (D_\beta D_\alpha)V^\mu \equiv [D_\alpha, D_\beta]V^\mu$ (1)

$$= D_\alpha (D_\beta V^\mu) - D_\beta (D_\alpha V^\mu)$$

↑ differentiates affine connection in D_β etc.

Thus: $[D_\alpha, D_\beta]V^\mu =$

$$\partial_\alpha (\partial_\beta V^\mu + \Gamma^\mu_{\lambda\beta} V^\lambda) + \Gamma^\mu_{\rho\alpha} (\partial_\beta V^\rho + \Gamma^\rho_{\lambda\beta} V^\lambda) - \Gamma^\sigma_{\beta\alpha} (\partial_\sigma V^\mu + \Gamma^\mu_{\lambda\sigma} V^\lambda)$$

$$- \partial_\beta (\partial_\alpha V^\mu + \Gamma^\mu_{\lambda\alpha} V^\lambda) - \Gamma^\mu_{\rho\beta} (\partial_\alpha V^\rho + \Gamma^\rho_{\lambda\alpha} V^\lambda) + \Gamma^\sigma_{\alpha\beta} (\partial_\sigma V^\mu + \Gamma^\mu_{\lambda\sigma} V^\lambda)$$

(4)

$$= (\partial_\alpha \Gamma^\mu_{\lambda\beta}) V^\lambda + \cancel{\Gamma^\mu_{\lambda\beta} \partial_\alpha V^\lambda} - (\partial_\beta \Gamma^\mu_{\lambda\alpha}) V^\lambda - \cancel{\Gamma^\mu_{\lambda\alpha} \partial_\beta V^\lambda}$$

$$+ \cancel{\Gamma^\mu_{\lambda\alpha} \partial_\beta V^\lambda} + \Gamma^\mu_{\rho\alpha} \Gamma^\rho_{\lambda\beta} V^\lambda - \cancel{\Gamma^\mu_{\lambda\beta} \partial_\alpha V^\lambda} - \Gamma^\mu_{\rho\beta} \Gamma^\rho_{\lambda\alpha} V^\lambda$$ (6)

Hence $[D_\alpha, D_\beta]V^\mu = \left\{ \partial_\alpha \Gamma^\mu_{\lambda\beta} - \partial_\beta \Gamma^\mu_{\lambda\alpha} + \Gamma^\mu_{\rho\alpha} \Gamma^\rho_{\lambda\beta} - \Gamma^\mu_{\rho\beta} \Gamma^\rho_{\lambda\alpha} \right\} V^\lambda$

$$\equiv R^\mu_{\lambda\beta\alpha} V^\lambda \neq 0 \text{ (in general)}$$ (7)

Riemann-Christoffel Curvature Tensor