ISOPERIMETRIC PROBLEMS

This is a class of problems where it is given to extremize one quantity subject to the constraint that another quantity remain fixed.

For example: A farmer with a fixed amount of fence material wants to enclose the maximum possible area for his horse to graze.

Formulation: We are given to extremize the integral I

$$I = \int_{x_1}^{x_2} dx\, f(x, y, y') \qquad y(x_1) = y_1 \quad y(x_2) = y_2 \qquad (1)$$

Subject to the constraint that some other integral J remains fixed:

$$J = \int_{x_1}^{x_2} dx\, g(x, y, y') = \text{constant} \qquad (2)$$

The solution to this problem requires LAGRANGE MULTIPLIERS which we review now.

Review of Lagrange Multipliers: [ARFKEN]

Consider the function $f(x, y, z)$ and evaluate

$$df = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z}\right)dz \qquad (3)$$

To find an extremum of f we set $df = 0$. Since the variations dx, dy, and dz are arbitrary, the only way that $df = 0$ can hold is if

$$\boxed{\partial f/\partial x = \partial f/\partial y = \partial f/\partial z = 0} \qquad (4)$$

Suppose now that we find that there is a constraint in the problem which can be expressed by some equation of the form

$$g(x, y, z) = 0 \qquad (5)$$

Because of this constraint, the variations dx, dy, dz are no longer independent, which was the assumption needed to derive the condition in (4). Specifically

$$g(x,y,z)=0 \Rightarrow 0=(\partial g/\partial x)dx + (\partial g/\partial y)dy + (\partial g/\partial z)dz \qquad (6)$$

Since $\partial g/\partial x$, $\partial g/\partial y$, and $\partial g/\partial z$ are known, one can solve (6) explicitly for dz, for example, in terms of dx and dy:

$$\underset{\sim}{dz} = -(\partial g/\partial z)^{-1}\left[(\partial g/\partial x)\underset{\sim}{dx} + (\partial g/\partial y)\underset{\sim}{dy}\right] \qquad (7)$$

Because of this equation, dz is dependent on dx and dy and the previous arguments to find the extremum are not valid.

One can of course eliminate dz simply by using (7) to replace dz everywhere. This can be done but is tedious.

There is another way to eliminate dz using Lagrange multipliers: Using (3) & (6) form the function $\boxed{f(x,y,z)+\underline{\lambda}\, g(x,y,z)}$. Then the extremum

$$df = 0 \qquad (8)$$

can be rewritten as $df + \lambda dg = 0$, since $g(x,y,z)=0 \Rightarrow dg=0$

This gives the following equation:

$$df(x,y,z) + \lambda\, dg(x,y,z) = 0 = \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial g}{\partial z}\right)dz \qquad (9)$$

Since dz (for example) is not linearly independent it should not appear in (9), and one way of ensuring this is to choose λ to make the coefficient of dz vanish:

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial g}{\partial z} = 0 \qquad (10)$$

Having eliminated dz, the expressions which give the extremum are now: $\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0$; $\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0$ (11)

When these equations are solved, $df = 0$ and $f(x,y,z)$ is an extremum subject to the constraint $g(x,y,z) = 0$.

$\boxed{\text{SUMMARY}}$:

- We want to find the extremum of $f(x,y,z)$ subject to the constraint $g(x,y,z) = 0$. Finding the extremum means finding x_0, y_0, z_0.

- Once we introduce the Lagrange multiplier λ, we then have <u>4 unknowns to solve for</u>: x_0, y_0, z_0, λ

- These 4 quantities are then determined by the following 4 equations

$$\left.\begin{aligned} &\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \quad (12a) \\ &\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \quad (12b) \end{aligned}\right\} \text{Eqs. (11) above}$$

$$\left.\frac{\partial f}{\partial z} + \lambda \frac{\partial g}{\partial z} = 0\right\} \text{Eq. (10)} \quad (12c)$$

$$\left.g(x,y,z) = 0\right\} \text{Eq. (5)} \quad (12d)$$

Example: Application of Lagrange Multipliers in QM

[ARFKEN] The ground state energy of a particle in rectangular QM box whose sides are a, b, c is given by $\left(E \equiv (4\pi^2)^{-1}\bar{E}\right)$ actual energy

$$E = E(a,b,c) = \frac{\hbar^2}{8m}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \tag{13}$$

We wish to find the shape of the box (i.e., a, b, c) such that E is a minimum for a fixed volume

$$V = V(a,b,c) = abc = \text{constant} \equiv k \tag{14}$$

Solution: In our previous notation let $f(a,b,c) = E(a,b,c)$ and

$$g(a,b,c) = V(a,b,c) - k = 0 = abc - k \tag{15}$$

We then solve:

$$\underbrace{\frac{\partial E}{\partial a} + \lambda\frac{\partial V}{\partial a} = 0}_{\text{this may be viewed as eliminating } da} \;;\; \frac{\partial E}{\partial b} + \lambda\frac{\partial V}{\partial b} = 0 \;;\; \frac{\partial E}{\partial c} + \lambda\frac{\partial V}{\partial c} = 0 \tag{16}$$

$$\frac{\partial E}{\partial a} + \lambda\frac{\partial V}{\partial a} = -\frac{\hbar^2}{4ma^3} + \lambda bc = 0 \tag{17a}$$

Similarly:

$$\frac{\partial E}{\partial b} + \lambda\frac{\partial V}{\partial b} = 0 = \frac{-\hbar^2}{4mb^3} + \lambda ac = 0 \tag{17b}$$

$$\frac{\partial E}{\partial c} + \lambda\frac{\partial V}{\partial c} = 0 = -\frac{\hbar^2}{4mc^3} + \lambda ab = 0 \tag{17c}$$

Multiplying this equations in turn by a, b, c then gives:

$$\lambda abc = \frac{\hbar^2}{4ma^2} \;;\; \lambda abc = \frac{\hbar^2}{4mb^2} \;;\; \lambda abc = \frac{\hbar^2}{4mc^2} \tag{18}$$

The solution to these equations is obviously $\boxed{a = b = c}$ (19)

$\Rightarrow$ rectangular box $\rightarrow$ cube

Note that we have solved the problem without having to actually determine λ. However, if we wish to solve for λ to give it a physical interpretation we can write:

$$\lambda abc = \frac{\hbar^2}{4ma^2} \xrightarrow{a=b=c} \lambda a^3 = \frac{\hbar^2}{4ma^2} \Rightarrow \boxed{\lambda = \frac{\hbar^2}{4ma^5}} \quad (20)$$

To interpret λ we note from (13) & (19) that

$$E = \frac{\hbar^2}{8m}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \rightarrow \frac{3\hbar^2}{8ma^2}$$

Hence the energy density is given by $\frac{E}{V} = \frac{(3/8)\hbar^2/ma^2}{a^3} = \frac{3}{8}\frac{\hbar^2}{ma^5}$

If we convert this to physical energy units, $\mathbf{E} = 4\pi^2 E$ etc then

$$\boxed{\lambda = \frac{3\pi^2}{2}\frac{E}{V}}$$

So λ is a measure of the energy density

EXAMPLE [2]: text p. 57

Extremize $f(x,y) = x^2+2xy$ subject to the constraint $x^2+y^2=4$

Solution: In this case the constraint is $x^2+y^2-4=0 \equiv g(x,y)$

From the preceding example & discussion, we want to extremize (i.e. minimize or maximize) the function $\mathcal{F} = f(x,y) - \lambda g(x,y)$

[Note: We have previously used $f+\lambda g$, whereas the text uses $f-\lambda g$. Either choice is purely conventional, since λ can itself be positive or negative]. Hence

$$\mathcal{F}(x,y) = f(x,y) - \lambda g(x,y) = x^2+2xy - \lambda(x^2+y^2-4) \quad (1)$$

Once λ is included we can now view the variations $\partial\mathcal{F}/\partial x$ and $\partial\mathcal{F}/\partial y$ are independent, so that

$$\frac{\partial \mathcal{F}(x,y)}{\partial x} = 2x+2y-2\lambda x = 0 \,;\; \frac{\partial \mathcal{F}(x,y)}{\partial y} = 2x - 2\lambda y = 0 \quad (2)$$

These two equations along with the original constraint equation $g = (x^2+y^2-4) = 0$ solve the problem as follows: From (2)

$$2x - 2\lambda y = 0 \Rightarrow \boxed{x = \lambda y} \quad (3)$$

Combining this with the first equation in (2) gives:

$$0 = 2x+2y-2\lambda x = \cancel{2}(\lambda \cancel{y}) + \cancel{2}\cancel{y} - \cancel{2}\lambda(\lambda \cancel{y}) = 0 \Rightarrow \lambda+1-\lambda^2 = 0 \quad (4)$$

$$2y(\lambda+1-\lambda^2) = 0$$

$$\boxed{\text{The solution of } \boxed{\lambda^2-\lambda-1=0} \text{ is } \lambda = \frac{1\pm\sqrt{5}}{2}} \quad (5)$$

Once we find $\lambda_\pm$ we can solve for the value(s) (x_0, y_0) where the extrema are.

Using $x = \lambda y$ (λ is either λ_+ or λ_-), and $x^2+y^2=4$ we get

$$x^2 = \lambda^2 y^2 \Rightarrow x^2+y^2 = \lambda^2 y^2 + y^2 = 4 \Rightarrow y^2 = \frac{4}{1+\lambda^2} \Rightarrow \boxed{y = \frac{\pm 2}{\sqrt{1+\lambda^2}}} \quad (6)$$

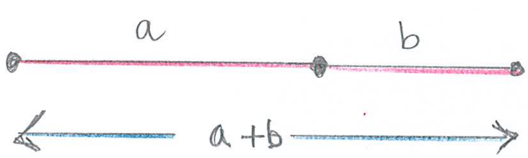
Lastly $x = \lambda y \Rightarrow$

$$\boxed{x = \frac{2\lambda}{\sqrt{1+\lambda^2}}} \quad (7)$$

SIDE COMMENT:

By design or not the solution $\boxed{\lambda_+ = \frac{1+\sqrt{5}}{2}}$ (8) GOLDEN RATIO plays an important role in art:

It is defined by



The GOLDEN RATIO is defined by the equation

$$\boxed{r = \frac{a}{b} = \frac{a+b}{a}} \quad (9) \Rightarrow \frac{a}{b} = 1 + \frac{b}{a} \Rightarrow r = 1 + \frac{1}{r} \Rightarrow \boxed{r^2 - r - 1 = 0} \quad (10)$$

Eq. (10) is the same as Eq. (5) above for λ. The solution λ_+ gives the GOLDEN RATIO: Numerically,

$$r = \frac{1+\sqrt{5}}{2} = 1.618033989\ldots \qquad (11)$$

THE MAXWELL – BOLTZMANN DISTRIBUTION

See GLASSTONE – "Theoretical Chemistry" p. 286 ff

This is simultaneously an application of permutations, combinations, Lagrange multipliers, and Stirling's Formula

Consider a gas containing n atoms or molecules. For simplicity we consider the case where the atoms are point objects described by coordinates $\vec{q}_j$ and momenta $\vec{p}_j$ $(j = 1, \ldots n)$. Each atom can be described by a point in phase space $(q_{j1}, q_{j2}, q_{j3}, p_{j1}, p_{j2}, p_{j3})$. If we suppress the index j temporarily then if q_1 is in the range q_1 to $q_1 + dq_1 \equiv (q_1 + \delta q_1)$, etc. then the volume δv in phase space occupied by this atom is

$$\delta v = \delta q_1 \delta q_2 \delta q_3 \, \delta p_1 \delta p_2 \delta p_3 \equiv \text{cell in phase space} \qquad (1)$$

This is the case for one atom or molecule.

More generally we can write $\delta v \longrightarrow \delta v_{i(j)}$ where the notation means that the jth molecule occupies the ith cell in phase space.

Combinatorics: The state of a collection of atoms can then be specified by asking how we can distribute the n atoms of a sample such that there are n_1 in cell δv_1, n_2 in cell δv_2, … etc. The number of ways this can be done $\equiv G$ can be found from our previous discussion. Clearly G is given by

$$\boxed{G = \frac{n!}{n_1!\, n_2! \ldots n_i!}} \qquad (2)$$

This is Eq. (5) on p. 153: $G \longleftrightarrow$ # of sentences; all atoms in the same cell are equivalent;

Stated another way: We are saying in a sense that the situation when n_1 words are $w_1, \ldots$ is the same as saying that n_1 atoms are in the phase-space volume δv_1.

Now the n atoms can be distributed in many ways such that $\sum_j n_j = n$. We then invoke one of the basic assumptions of statistical mechanics that the probability W that the system will be in a certain configuration (specified by $\{n_j\}$) is proportional to G computed for that configuration:

$$W = \text{Constant} \otimes G \equiv cG \qquad (3)$$

The most likely configuration is then the one that maximizes W.

To find out what configuration this is consider $\ln W$

$$\ln W = \ln c + \ln n! - \sum_i \ln (n_i!) \qquad (4)$$

Use the Stirling Formula: $\boxed{n! \cong \sqrt{2\pi n}\; n^n e^{-n}}$ (5)

Approximately then, $\ln(n!) \cong \ln\sqrt{2\pi} + \ln(n^{1/2}) + n\ln n - n$ (6)

$$\therefore \boxed{\ln(n!) \cong n\ln n - n} \qquad (7)$$

Hence: $\ln W \cong \ln c + (n\ln n - n) - \sum_i (n_i \ln n_i - n_i)$

$$= \ln c + (n\ln n - \cancel{n}) - \sum_i n_i \ln n_i + \underbrace{\cancel{\sum_i n_i}}_{n} \qquad (8)$$

Hence: $\boxed{\ln W \cong \ln c + n\ln n - \sum_i n_i \ln n_i}$ (9)

We want to extremize (here maximize) W as a function of the n_i

So we consider:

$$\delta(\ln W) = \underset{0}{\underbrace{\delta(\ln c)}} + \underset{\substack{0 \\ (n = \text{fixed})}}{\underbrace{\delta(n\ln n)}} - \delta\left(\sum_i n_i \ln n_i\right) \qquad (10)$$

It follows from (10) that

$$0 \equiv \delta(\ln W) = -\sum_i \left(n_i \frac{1}{n_i} + \ln n_i \right) \delta n_i \Rightarrow \sum_i (1 + \ln n_i)\delta n_i = 0 \quad (11)$$

↳ We wish to carry out this extremization subject to the constraints:

$$\sum_i n_i = n \Rightarrow \alpha \sum_i \delta n_i = 0 \quad ; \; n = \text{total \# of atoms}$$

$$\sum_i \epsilon_i n_i = E \Rightarrow \beta \sum_i \epsilon_i \delta n_i = 0 \quad E = \text{total energy of atoms}$$

(12)

α, β are Lagrange multipliers

Combining (11) & (12) we find: $\sum_i (1 + \ln n_i + \alpha + \beta \epsilon_i)\delta n_i = 0$

Having introduced α and β we can now argue that the δn_i are arbitrary since they are not constrained any longer by (12). Since the δn_i are arbitrary we can then argue that the coefficient of each δn_i must separately vanish. This gives:

$$1 + \ln n_i + \alpha + \beta \epsilon_i = 0 \Rightarrow \ln n_i = \underbrace{(-1-\alpha)}_{\text{another constant-}\alpha' \to \text{ rename as } -\alpha} - \beta \epsilon_i \quad (13)$$

sign is conventional

$$\therefore \ln n_i = -\alpha - \beta \epsilon_i$$

$$\Rightarrow n_i = \underbrace{e^{-\alpha}}_{\text{normalization constant}} e^{-\beta \epsilon_i} \Rightarrow \boxed{n_i(\epsilon_i) = \text{Const } e^{-\beta \epsilon_i}} \quad (14)$$

By comparing to the usual macroscopic ideal gas laws we can then deduce that $\beta = 1/k_B T$.* So finally

$$\boxed{n_i(\epsilon_i) = \text{Const } e^{-\epsilon_i / k_B T}} \quad (15)$$

MAXWELL-BOLTZMANN DISTRIBUTION

* SEE GLASSTONE p. 295

DEFINITE INTEGRALS IN SEVERAL VARIABLES

Previously we have considered integrals over one variable of the form:

$$I = \int_{x_1}^{x_2} f(x)\, dx \qquad (1)$$

This integral is evaluated over a one-dimensional path connecting x_1 and x_2. Now we want to generalize this to several variables beginning with a 2-dimensional problem. So we consider a function $f(x,y)$ and integrate this over a 2-dimensional domain D:

$$\boxed{F(D) = \int_{y_1}^{y_2} \left[\int_{x_1(y)}^{x_2(y)} f(x,y)\, dx \right] dy} \qquad (2) \quad \Big\} \text{ y is held constant during the x integration}$$

In (2) the expression in [⋯] defines a function we may call $g(y)$ and so the final result is

$$F(D) = \int_{y_1}^{y_2} g(y)\, dy \qquad (3)$$

Of course, the roles of x and y are interchangeable. The net result is that the final result can be reduced to a sequence of 1-dimensional integrations.

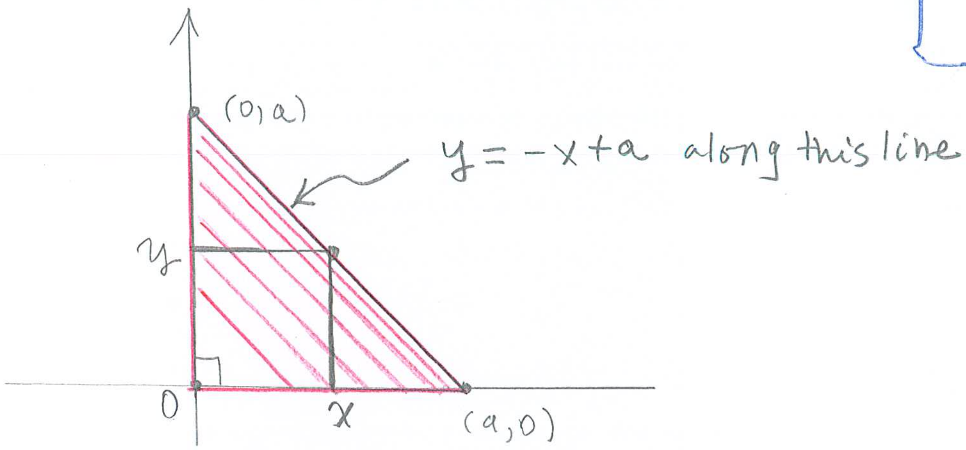
Let's begin with a simple example: Suppose that $f(x,y) = X(x)\, Y(y)$

Then:

$$I(D) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} X(x)\, Y(y)\, dx\, dy = \int_{y_1}^{y_2} Y(y)\, dy \int_{x_1}^{x_2} X(x)\, dx \qquad (4)$$

This is simple because the limits x_1, x_2 do not depend on y_1, y_2

EXAMPLE [2] – {Sherwood/Taylor CALCULUS p. 514}



This figure shows a triangular slab set down at the origin, and in the shape of an isosceles right triangle with sides a. It is given that the density varies as the square of the distance from the right angle, located at the origin. The problem is to find its mass.

Solution: The mass M is given by the integral of the density over the area of the triangle:

$$M = \iint_A \rho(x,y)\,dy\,dx \qquad \rho = \text{density} \qquad (5)$$

We are given that

$$\rho(x,y) = c(x^2+y^2) \qquad c = \text{constant} \qquad (6)$$

Then $M = c\iint_A (x^2+y^2)\,dx\,dy$ or $c\iint_A (x^2+y^2)\,dy\,dx$ (7)

Since this problem is symmetric between x and y we can arbitrarily choose to integrate first over y, keeping x fixed. We note from the figure that the outer boundary of the Δ slab is given by the line $y = -x+a$, as can be checked by finding the equation of the line which passes through the two points (a,0) and (0,a).

It then follows from the figure that when we carry out $\int dy$ keeping x fixed, y ranges between $y=0$ and $y=a-x$. This is an example of the more typical situation in which the integration is over some domain D (here a triangle) which constrains the variables of integration such that the limits on one integration variable depend on the other integration variable. It now follows that

$$M = c\int_0^a dx \underbrace{\int_0^{a-x} (x^2+y^2)\,dy}_{I(x,a)} \; ; \; I(x,a) = \left[x^2 y + \frac{y^3}{3} \right]_{y=0}^{y=a-x} \qquad (8)$$

$$\Rightarrow I(x,a) = x^2(a-x) + \frac{1}{3}(a-x)^3 = \frac{1}{3}a^3 - a^2x + 2ax^2 - \frac{4}{3}x^3 \qquad (9)$$

(Note that there is no contribution from the lower limit!)

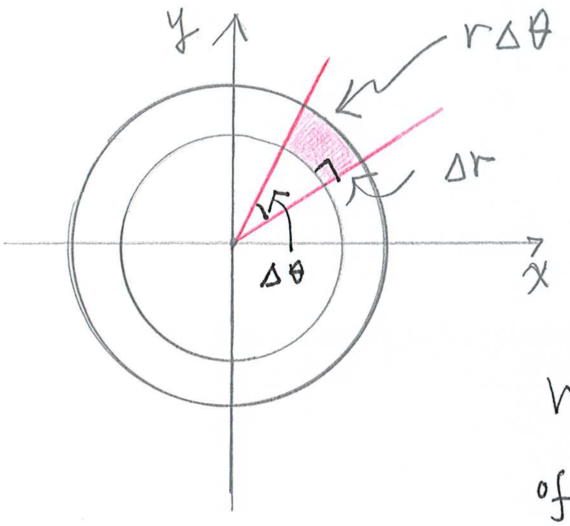
Next we evaluate $M = c\int_0^a I(x,a)\,dx$

$$\Rightarrow \boxed{M = c\left\{ \frac{1}{3}a^3x - \frac{1}{2}a^2x^2 + \frac{2}{3}ax^3 + \frac{1}{3}x^4 \right\}_0^{x=a} = \frac{1}{6}ca^4} \qquad (10)$$

CHOOSING COORDINATE SYSTEMS FOR MULTIPLE INTEGRALS

Integrals over some domain D can be simplified by choosing variables which reflect the underlying symmetry of D. This is particularly true for domains which have circular, cylindrical, or spherical symmetry.

Suppose we wish to find the formula for the area of a circle:



We go from $(x,y) \to (r,\theta)$ via

$$x = r\cos\theta \quad y = r\sin\theta \qquad (1)$$

or

$$r = \sqrt{x^2+y^2} \quad ; \theta = \tan^{-1}\frac{y}{x} \qquad (2)$$

We see from the figure that in the limit of very small patches the changes Δr in r and $\Delta\theta$ in θ are $\perp$ to eachother. It follows, that in this limit the shaded area is approximately a rectangle whose area is $dA \doteq \Delta A \approx (\Delta r)(r\Delta\theta)$. Hence in polar coordinates we are adding up (or integrating) infinitesimal rectangles of area

$$\boxed{dA = dr \cdot r\,d\theta = r\,dr \cdot d\theta.}$$

Hence the area of a circle of radius R is given by

$$A = \underbrace{\int_{r=0}^{R} r\,dr}_{\frac{1}{2}R^2} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} = \frac{1}{2}R^2 \cdot 2\pi = \pi R^2 \qquad (3)$$

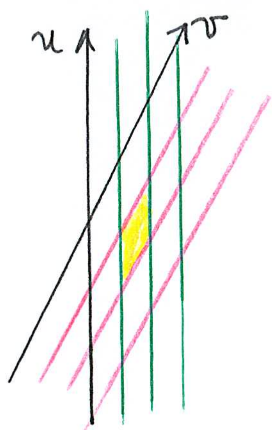
ORTHOGONAL COORDINATES AND JACOBIANS

We can extract from the previous example 2 lessons:

[1] An appropriate choice of coordinate system can significantly simplify a calculation, especially if it conforms to the symmetry of a problem

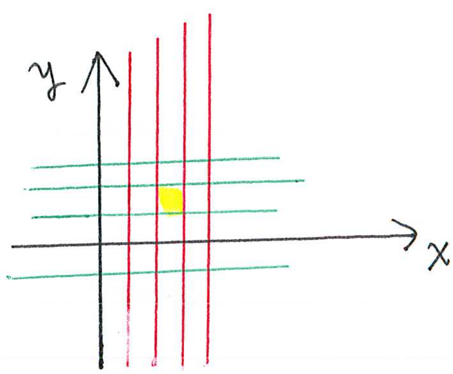
[2] We always want to work in a coordinate system where the new coordinates $\xi_1, \xi_2, \dots$ have the property that curves of constant ξ_1 are orthonal (i.e. perpendicular) to curves of constant $\xi_2, \dots$ etc. This allows us to view infinitesimal volumes as rectangles, which then simplifies various calculations.

The following is not an orthogonal coordinate system, since even at the microscopic scale the infinitesimal elements of area are not rectangles:



here each infinitesimal area element is a parallelogram, even as the area $\rightarrow 0$

Contrast the u-v system above, to the usual x-y coordinate system below:



The JACOBIAN:

From the previous discussion, when we wish to find the area of some 2-dimensional figure, we can start with rectangular coordinates in which case

$$A = \int dA = \int dx\,dy = \int 1 \cdot dx\,dy \qquad (1)$$

However, we could also start with polar coordinates in which case

$$A = \int dA = \int r \cdot dr\,d\theta \qquad (2)$$

More generally, in any coordinate system u_1, u_2 we can write the area as

$$A = \int J \cdot du_1\,du_2 \equiv \int J \cdot du\,dv \qquad (3)$$

J is called the Jacobian and can be thought of as normalizing the area element dA relative to Cartesian coordinates for which $J=1$. J is then obtained by evaluating the determinant of the matrix:

$$J = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \qquad (4)$$

For the transition from Cartesian to polar coordinates we have

$$x = r\cos\theta \qquad y = r\sin\theta \qquad (5)$$

$$\frac{\partial x}{\partial r} = \cos\theta \;;\; \frac{\partial x}{\partial \theta} = -r\sin\theta \;;\; \frac{\partial y}{\partial r} = \sin\theta \;;\; \frac{\partial y}{\partial \theta} = r\cos\theta \qquad (6)$$

$$J = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \frac{\partial x}{\partial r} \cdot \frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial r} \cdot \frac{\partial x}{\partial \theta} = (\cos\theta)(r\cos\theta) - (\sin\theta)(-r\sin\theta) \qquad (7)$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$\boxed{\Rightarrow \int dA = \int J \cdot dr\,d\theta = \int r \cdot dr\,d\theta} \qquad (8)$$

The JACOBIAN (continued):

Notation: To keep track of the derivatives we are computing we use the shorthand

$$J = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} \equiv \frac{\partial(x,y)}{\partial(r,\theta)} \qquad (9)$$

In 3-dimensions the volume of an object is given by

$$V = \int dV = \int 1\, dx\, dy\, dz = \int J\, dr\, d\theta\, d\phi \qquad (10)$$

Here we transform to spherical coordinates via (see text p.67)

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \qquad (11)$$

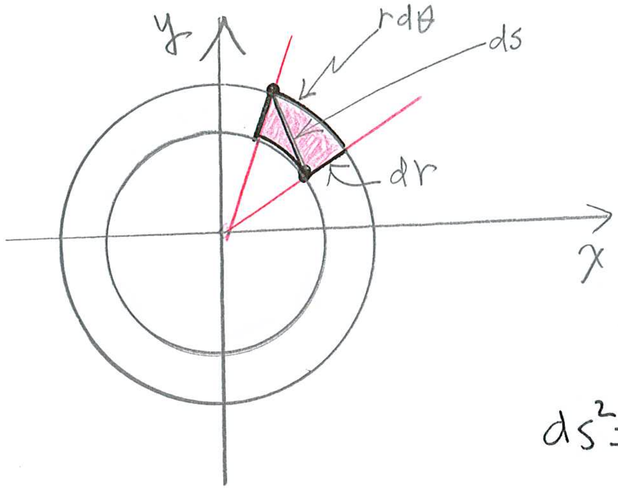
Then

$$J \equiv \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \det\begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix} \qquad (12)$$

$$\Rightarrow \boxed{J = r^2\sin\theta \qquad \Rightarrow \int dV = \int r^2\sin\theta \cdot dr\, d\theta\, d\phi = \int r^2 dr \cdot \sin\theta\, d\theta\, d\phi} \qquad (13)$$

The JACOBIAN (Continued):

Here is a shortcut for calculating J using a little bit of tensor analysis:



In this shaded region, which will eventually be approximated by an infinitesimal rectangle, the length ds of the diagonal is given by the Pythagorean Theorem:

$$ds^2 = dr^2 + r^2 d\theta^2 \equiv g_{rr} dr^2 + g_{\theta\theta} d\theta^2 \qquad (9)$$

Then $J = \sqrt{g} \equiv \sqrt{g_{rr} \cdot g_{\theta\theta}} = \sqrt{1 \cdot r^2} = r$ ✓ $\qquad (10)$

In 3-dimensions the analogous expression is given by:

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$
$$\equiv g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2$$

$$J = \sqrt{g} \equiv \sqrt{g_{rr} \cdot g_{\theta\theta} \cdot g_{\phi\phi}} = \sqrt{1 \cdot r^2 \cdot r^2 \sin^2\theta} = r^2 \sin\theta$$

$$\Rightarrow \int dV = \int J \, dr \, d\theta \, d\phi = \int r^2 \sin\theta \, dr \, d\theta \, d\phi$$
$$= \int r^2 dr \cdot \sin\theta \, d\theta \cdot d\phi \ ✓✓$$

NOTE: This shortcut, and its analogs in higher dimensions, only works with orthogonal coordinates, which allow the use of the Pythagorean Theorem.