(5i) This can be proved by noting that $\delta[g(x)]$ will differ from zero only when $g(x)=0$ which means that this holds for values $x=x_i$ which are the roots of $g(x)$: $g(x_i)=0$.

Hence in the vicinity of each root we can expand $g(x)$ as

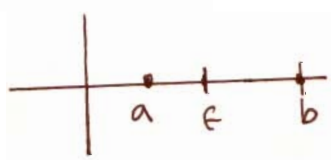
$$g(x) = \underbrace{g(x_i)}_{0} + (x-x_i)\left.\frac{dg}{dx}\right|_{x_i} + \frac{1}{2}(x-x_i)^2 \left.\frac{d^2g}{dx^2}\right|_{x_i} + \cdots \quad (8)$$

$$= (x-x_i)\left\{\left.\frac{dg}{dx}\right|_{x_i} + \underbrace{\frac{1}{2}(x-x_i)\left.\frac{d^2g}{dx^2}\right|_{x_i} + \cdots}_{0}\right\} \cong (x-x_i)\underbrace{\left.\frac{dg}{dx}\right|_{x_i}}_{\text{constant} \equiv a} \quad (9)$$

Hence near a root x_i: $\delta[g(x)] \cong \delta\left[\left.\frac{dg}{dx}\right|_{x_i}(x-x_i)\right]$

$$= \frac{1}{|\partial g/\partial x|_{x_i}|}\,\delta(x-x_i) \leftarrow \text{using 5(e)} \quad (10)$$

Since we can repeat this process for each root, we sum over all the roots. This can be seen from the following example: Consider the function $g(x)=(x-a)(x-b)$ with roots at $x=a$ and $x=b$. Given $g(x)$ we have



$$\int_{-\infty}^{\infty} dx\,\delta[g(x)]f(x) = \underbrace{\int_{-\infty}^{\epsilon} dx\,\delta[(x-a)(x-b)]f(x)}_{\mathrm{I}} + \underbrace{\int_{\epsilon}^{+\infty} dx\,\delta[(x-a)(x-b)]f(x)}_{\mathrm{II}} \quad (11)$$

Near $x=a$ I gives: $\mathrm{I} \cong f(a)\int_{-\infty}^{\epsilon} dx\,\delta[(x-a)(a-b)] = \frac{f(a)}{|a-b|}\underbrace{\int_{-\infty}^{\epsilon} dx\,\delta(x-a)}_{1}$ (12)

$$= \frac{1}{|a-b|}f(a)$$

Near $x=b$ II gives: $\mathrm{II} \cong f(b)\int_{\epsilon}^{\infty} dx\,\delta[(b-a)(x-b)] = \frac{f(b)}{|b-a|}$ (13)

Combining the results in (11)–(13) we have

$$\int_{-\infty}^{\infty} dx\, \delta[g(x)]\, f(x) = \frac{1}{|a-b|} f(a) + \frac{1}{|b-a|} f(b) = \frac{1}{|a-b|}\left[f(a) + f(b)\right] \qquad (14)$$

Compare this to the result using the formula in (5i):

$$dg/dx = d/dx\left(x^2 - (a+b)x + ab\right) = 2x - (a+b) \qquad (15)$$

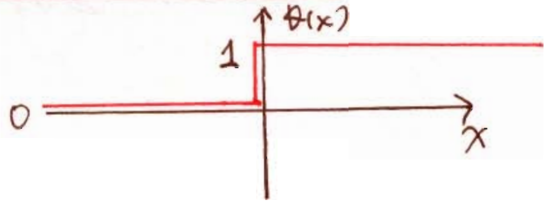
$$\partial g/\partial x\big|_{x=a} = 2a - (a+b) = a-b$$

$$\partial g/\partial x\big|_{x=b} = 2b - (a-b) = b-a \qquad (16)$$

Hence $$\delta[g(x)] = \sum_i \frac{1}{|\partial g/\partial x|_{x_i}}\, \delta(x-x_i) = \frac{1}{|a-b|}\delta(x-a) + \frac{1}{|b-a|}\delta(x-b) \qquad (17)$$

and this clearly reproduces (14) above. ✓

The Step Function $\theta(x)$:



$\theta(x) = 1 \; ; x > 0$

$\theta(x) = 0 \; ; x < 0$

$\theta(0) \equiv 1/2$

Claim: $\quad \dfrac{d}{dx}\theta(x) = \delta(x) \qquad (1)$

Proof: Consider $I = \int_{-\infty}^{\infty} dx \frac{d}{dx}\theta(x) f(x)$ where $f(\pm\infty) = 0$

Then $\displaystyle \int_{-\infty}^{\infty} dx \underbrace{\left[\frac{d}{dx}\theta(x)\right]}_{dv} \underbrace{f(x)}_{u} = \theta(x) f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\, \theta(x) \frac{d}{dx} f(x) \qquad (2)$

$\displaystyle = -\int_{0}^{\infty} \cancel{dx} \frac{d}{\cancel{dx}} f(x) = -\int_{0}^{\infty} d f(x) = -\Big[\underbrace{f(\infty)}_{0} - f(0)\Big] = +f(0) \qquad (3)$

Comparing in (2) & (3) we see that $\left[\frac{d}{dx}\theta(x)\right]$ has the same effect as $\delta(x)$. ✓

SPECIFIC REPRESENTATIONS OF $\delta(x)$:

As noted previously, $\delta(x)$ is not a conventional mathematical function. Rather it can be viewed as the limiting case of a function whose width decreases as its height increases (when some parameter is varied) in such a way that its area remains $= 1$.

We present several examples:

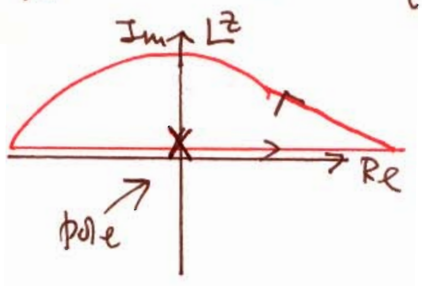
(A) $\boxed{f_a(x) \equiv \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}}$; $\int_{-\infty}^{\infty} dx\, f_a(x) = 1$ independent of a (1)

Then $\boxed{\delta(x) = \lim_{a\to 0} f_a(x) = \lim_{a\to 0} \frac{1}{a\sqrt{\pi}} e^{-x^2/a^2}}$ (2)

Here we note that as $a \to 0$ $e^{-x^2/a^2} \to 0$ for $x \neq 0$; moreover $e^{-x^2/a^2} \to 0$ faster than $1/a \to \infty$. Hence $f_a(x\neq 0) = 0$ as $a \to 0$. However, as $a \to 0$ $f_a(0) \sim \infty$ to keep the area constant.

(B) $\boxed{h_g(x) = \frac{\sin gx}{\pi x}}$ (3) This can be integrated using Contour integration (see end of semester!)

$$\int_{-\infty}^{\infty} dx\, h_g(x) = \mathrm{Im}\, \frac{1}{\pi} \int_{-\infty}^{\infty} dx\, \frac{1}{x} e^{igx} = \frac{1}{\pi} \mathrm{Im} \left\{ \pi i \left[e^{igx} \right]_{x=0} \right\} = 1 \quad (4)$$



We note that for $x \cong 0$ $h_g(x) \cong \frac{g}{\pi}$; Hence as $g \to \infty$ $h_g(x \cong 0) \to \infty$

Since $\int_{-\infty}^{\infty} dx\, h_g(x) = 1$ (for all values of g) it follows that | 28, 29

the remaining contributions for $x \neq 0$ are becoming vanishingly small.
This happens because $\sin(gx)$ oscillates very rapidly as $g \to \infty$
(this is the Riemann - Lebesgue Theorem). We can thus finally
write:

$$\boxed{\delta(x) = \lim_{g\to\infty} \frac{\sin gx}{\pi x}} \quad (5)$$

(c) The 3RD representation that we consider is

$$\boxed{\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2}} \quad (6)$$

Clearly the r.h.s. of (6) vanishes as $\epsilon \to 0$ for all $x \neq 0$. For $x=0$ the r.h.s. $\to 1/\epsilon$ as $\epsilon \to 0$, so (6) has the correct behavior.

Note that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = \lim_{\epsilon\to 0} \frac{\epsilon}{\pi} \int_{-\infty}^{+\infty} dx \frac{1}{x^2+\epsilon^2} = \lim_{\epsilon\to 0} \frac{\epsilon}{\pi} \cdot \frac{1}{\epsilon} \tan^{-1}\frac{x}{\epsilon}\Big|_{-\infty}^{\infty}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1 \;; \quad \text{independent of } \epsilon \quad (7)$$

Hence the function in (6) also has unit area (independent of ϵ), and as $\epsilon \to 0$ this function vanishes everywhere except at $x=0$.
From the previous discussion this establishes that (6) is a valid representation of $\delta(x)$.

Comments on Representations of $\delta(x)$:

Here we evaluate some of the integrals we discussed previously.

Consider

$$I = \int_{-\infty}^{\infty} dx\, e^{-x^2} \Rightarrow I^2 = \int_{-\infty}^{\infty} dx\, e^{-x^2} \int_{-\infty}^{\infty} dy\, e^{-y^2} \qquad (1)$$

$$\text{Hence } I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\; e^{-(x^2+y^2)} \qquad (2)$$

Transforming to polar coordinates: $dx\,dy \to 2\pi r\,dr \qquad x^2+y^2=r^2$

$$\therefore\ I^2 = 2\pi \int_0^{\infty} dr\,.\,r e^{-r^2} \xrightarrow[\rho=r^2]{} 2\pi\cdot\frac{1}{2}\int_0^{\infty} d\rho\, e^{-\rho} = -\pi e^{-\rho}\Big|_0^{\infty} = \pi \qquad (3)$$

$$\Rightarrow d\rho = 2r\,dr$$

$$\text{Hence } I^2 = \pi \Rightarrow \boxed{I = \int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}} \qquad (4)$$

It follows from (4) that $\int_{-\infty}^{\infty} dy\, e^{-y^2/a^2} = a\int_{-\infty}^{\infty} dx\, e^{-x^2} = a\sqrt{\pi}$ (5)

Hence

$$\boxed{\frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} dy\, e^{-y^2/a^2} = 1\ ;\ \underline{\text{independent of } a}} \qquad (6)$$

Other related integrals can be evaluated in a similar way: Consider

$$I^3 = (\sqrt{\pi})^3 = \iiint_{-\infty}^{\infty} dx\,dy\,dz\; e^{-(x^2+y^2+z^2)} = 4\pi\int_0^{\infty} dr\,.\,r^2 e^{-r^2} \qquad (7)$$

$$\text{Hence } \boxed{\int_0^{\infty} dr\,.\,r^2 e^{-r^2} = \frac{\sqrt{\pi}}{4}} \qquad (8)$$

Another way to derive Eq.(8) is to start with Eq.(6) and let $b = 1/a^2$. Then

$$f(b) \equiv \int_{-\infty}^{\infty} dy\, e^{-by^2} = \sqrt{\frac{\pi}{b}} \qquad (9)$$

$$\frac{df(b)}{db} = -\int_{-\infty}^{\infty} dy \cdot y^2 e^{-by^2} = \frac{d}{db}\left(\sqrt{\frac{\pi}{b}}\right) = -\frac{1}{2}\sqrt{\frac{\pi}{b^3}} \qquad (10)$$

Combining (9) & (10) we find: $\boxed{\int_0^{\infty} dy\, y^2 e^{-by^2} = \frac{1}{2}\int_{-\infty}^{\infty} dy \ldots = \frac{1}{4}\sqrt{\frac{\pi}{b^3}}} \qquad (11)$

Setting $b=1$ in (11) then leads immediately to (8). ✓

KEY THEOREM IN POTENTIAL THEORY:

If the divergence and curl of a vector field $\vec{V}(r)$ are known,

$$\vec{\nabla}\cdot\vec{V}(\vec{r}) = s(\vec{r}) \quad \sim \text{charges} \qquad (1a)$$

$$\vec{\nabla}\times\vec{V}(\vec{r}) = \vec{c}(\vec{r}) \quad \sim \text{currents} \qquad (1b)$$

throughout space, and if there are no sources or currents at ∞ $[\,s(\infty)=0 \quad \vec{c}(\infty)=0\,]$ then $\vec{V}(\vec{r})$ is uniquely given by

$$\boxed{\vec{V}(\vec{r}) = -\vec{\nabla}\phi(\vec{r}) + \vec{\nabla}\times\vec{A}(\vec{r})} \qquad (2)$$

where $(\vec{x}\equiv\vec{r})$

$$\phi(\vec{x}) = \frac{1}{4\pi}\int d^3x' \, \frac{s(\vec{x}\,')}{r(\vec{x},\vec{x}\,')} \qquad (3)$$

$$\vec{A}(\vec{x}) = \frac{1}{4\pi}\int d^3x' \, \frac{\vec{c}(\vec{x}\,')}{r(\vec{x},\vec{x}\,')} \quad ; r(\vec{x},\vec{x}\,') = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \qquad (4)$$

<u>Proof:</u> We first show that (2) is a solution, and then show that it is the unique solution.

Consider first $$\vec{\nabla}\cdot\vec{V}(\vec{x}) = -\nabla^2\phi(\vec{x}) + \underbrace{\vec{\nabla}\cdot[\vec{\nabla}\times\vec{A}(\vec{x})]}_{=0 \text{ using } 17(8)} \qquad (5)$$

Hence $$\vec{\nabla}\cdot\vec{V}(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \, \nabla^2_{(x)}\left\{\frac{s(\vec{x}\,')}{r(\vec{x},\vec{x}\,')}\right\} \qquad (6)$$

this means that ∇^2 acts only on $\vec{x}$ and not on $\vec{x}\,'$ in $\{\ldots\}$

Then

$$\nabla^2\left(\frac{1}{r}\right) = \nabla^2\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right) = -4\pi\,\delta^3(\vec{r}) = -4\pi\,\delta^3(\vec{x}-\vec{x}\,') \qquad (7)$$

20.1(1)

Hence $\vec{\nabla}\cdot\vec{V}(\vec{x}) = -\nabla^2\phi(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \left\{-4\pi\delta^3(\vec{x}-\vec{x}')\,S(\vec{x}')\right\}$ (8)

$$\boxed{\therefore\ \vec{\nabla}\cdot\vec{V}(\vec{x}) = S(\vec{x})\ \checkmark} \quad (9)$$

This establishes that Eq.(2) is indeed the solution to Eq. (1a).

$\#$

We next show that Eq.(2) is also the solution to Eq. (1b). This requires some more effort, but allows us to gain some practice manipulating ∇^2, $\vec{\nabla}\times$, ...

From (2) we have: $\vec{\nabla}\times\vec{V} = \vec{\nabla}\times\left\{-\vec{\nabla}\phi + \vec{\nabla}\times\vec{A}\right\}$ (10)

$$= \underbrace{-\vec{\nabla}\times(\vec{\nabla}\phi)}_{0} + \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) \underset{\uparrow\, 23(17)}{=} -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$$

Hence $\vec{\nabla}\times\vec{V}(\vec{x}) = -\frac{1}{4\pi}\int d^3x'\, \nabla^2_{(x)}\left\{\frac{\vec{c}(\vec{x}')}{r(\vec{x},\vec{x}')}\right\}$ (11) → I

$$+ \frac{1}{4\pi}\int d^3x'\, \vec{\nabla}_{(x)}\left\{\vec{\nabla}_{(x)}\cdot\left(\frac{\vec{c}(\vec{x}')}{r(\vec{x},\vec{x}')}\right)\right\}$$

→ II

We will later show that II $=0$. Assuming this for now we can directly repeat the steps leading to (9) which then give from I

$$\vec{\nabla}\times\vec{V}(\vec{x}) = -\frac{1}{4\pi}\int d^3x' \left[\nabla^2_{(x)}\left(\frac{1}{r}\right)\right]\vec{c}(\vec{x}') = -\frac{1}{4\pi}\int d^3x' \left(-4\pi\delta^3(\vec{x}-\vec{x}')\right)\vec{c}(\vec{x}') \quad (12)$$

$$\therefore\ \boxed{\vec{\nabla}\times\vec{V}(x) = c(\vec{x})\ \checkmark} \quad (13)$$

This establishes that (2) is also a solution of Eq. (1b), provided that we can now show that (II) = 0. 36,37

Define (II) $\equiv \vec{D} = \frac{1}{4\pi}\int d^3x' \, \vec{\nabla}_{(x)} \left[\vec{\nabla}_{(x)} \cdot \frac{\vec{C}(\vec{x}')}{r(\vec{x},\vec{x}')} \right]$ (14)

To clarify the following steps we insert subscripts on $\vec{\nabla}$ so that we can keep track of them. Both $\vec{\nabla}_{(x)}$ operators only operate on $\vec{x}$:

$$\vec{D} = \frac{1}{4\pi}\int d^3x' \, \vec{\nabla}_1 \left[\vec{\nabla}_2 \cdot \left(\frac{\vec{C}}{r}\right)\right] = \frac{1}{4\pi}\int d^3x' \, \vec{\nabla}_1 \left[\vec{C}(\vec{x}') \cdot \vec{\nabla}_2 \left(\frac{1}{r}\right)\right] \quad (15)$$

$$= \frac{1}{4\pi}\int d^3x' \, (\vec{C}\cdot\vec{\nabla}_2)(\vec{\nabla}_1(1/r)) \equiv \frac{1}{4\pi}\int d^3x' \, (\vec{C}\cdot\vec{\nabla})(\vec{\nabla}(1/r)) \quad (16)$$

Note here that both $\vec{\nabla}$ operators only act on $1/r = 1/r(\vec{x},\vec{x}')$, since $1/r$ contains the only dependence on $\vec{x}$. This can be made clearer if we write $\vec{D}$ in the form

$$\vec{D} = \frac{1}{4\pi}\int d^3x' \left[\vec{C}(\vec{x}') \cdot \vec{\nabla}_{(x)}\right] \left[\vec{\nabla}_{(x)} \left(\frac{1}{r(\vec{x},\vec{x}')}\right)\right] \quad (17)$$

We next introduce the following <u>trick</u>: First we now will denote

$$\vec{\nabla}_{(x)} = \hat{\imath}\frac{\partial}{\partial x} + \cdots + \hat{k}\frac{\partial}{\partial z} \equiv \vec{\nabla} \quad (18)$$

$$\vec{\nabla}' = \hat{\imath}\frac{\partial}{\partial x'} + \hat{\jmath}\frac{\partial}{\partial y'} + \hat{k}\frac{\partial}{\partial z'} \quad (19)$$

Trick: $$\boxed{\vec{\nabla}' g[r(\vec{x},\vec{x}')] = -\vec{\nabla} g[r(\vec{x},\vec{x}')]} \quad (20)$$

Example: Let $g(r) = \frac{1}{2}c\,r^2 = \frac{1}{2}c\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]$ (21)