# Physics Graduate School Qualifying Examination

## Fall 2017 Part I

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**Instructions:** Work all problems. This is a closed book examination. Calculators may not be used. **Start each problem on a new pack of correspondingly numbered paper and use only one side of each sheet. Place your 3-digit identification number in the upper right hand corner of each and every answer sheet. All sheets, which you receive, should be handed in, even if blank.** Your 3-digit ID number is located on your envelope. All problems carry the same weight.

- Correct answers without adequate explanation/reasoning will not get full credit.
- Explain *all* variables you use in your derivations.
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- Use correct vector notation when appropriate.
- Your work must be legible and clear.

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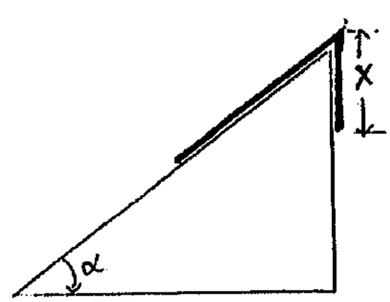
1) An ideal, uniform rope (no friction) with total mass M and total length L rests on a slanted plane as shown below:



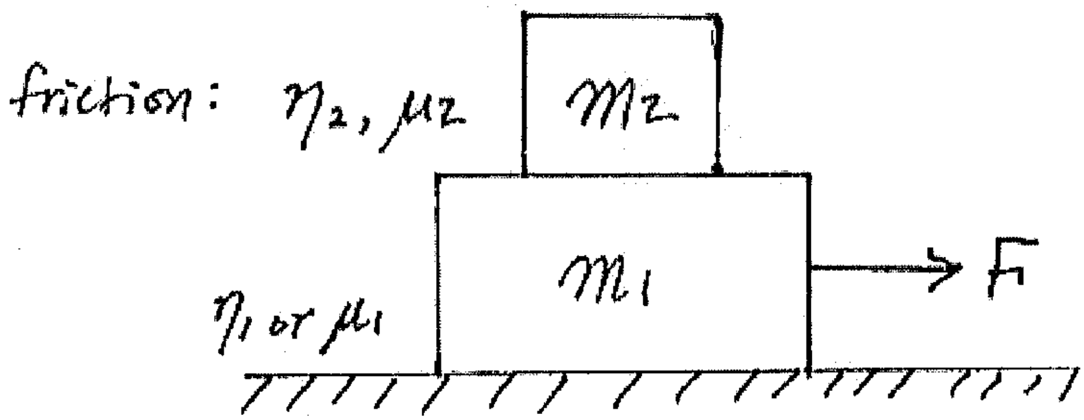

a) How long is the section *x*, if the rope does not move?

b) Consider the acting forces, and derive the equation of motion for x.

Take the gravitational acceleration to be g.

2) A block with mass $m_2$ is placed on top of another block with mass $m_1$ as shown below. A force F is applied to the right on the lower block. The coefficient of kinetic friction between the upper block and the lower block is $\mu_2$ and the coefficient of kinetic friction between the lower block and the floor is $\mu_1$. Similarly, the static friction coefficients are $\eta_1$ and $\eta_2$, respectively. Make sure to indicate the direction whenever necessary in answering the questions below. The positive direction is to the right.

(a) If the upper block is slipping against the lower block, what is the acceleration of the upper block? You will need to indicate both the magnitude and direction.
(b) If the upper block is slipping against the lower block, what is the acceleration of the lower block?
(c) What is the required condition that the upper block does not slip against the lower block? Assume that the lower block is moving.

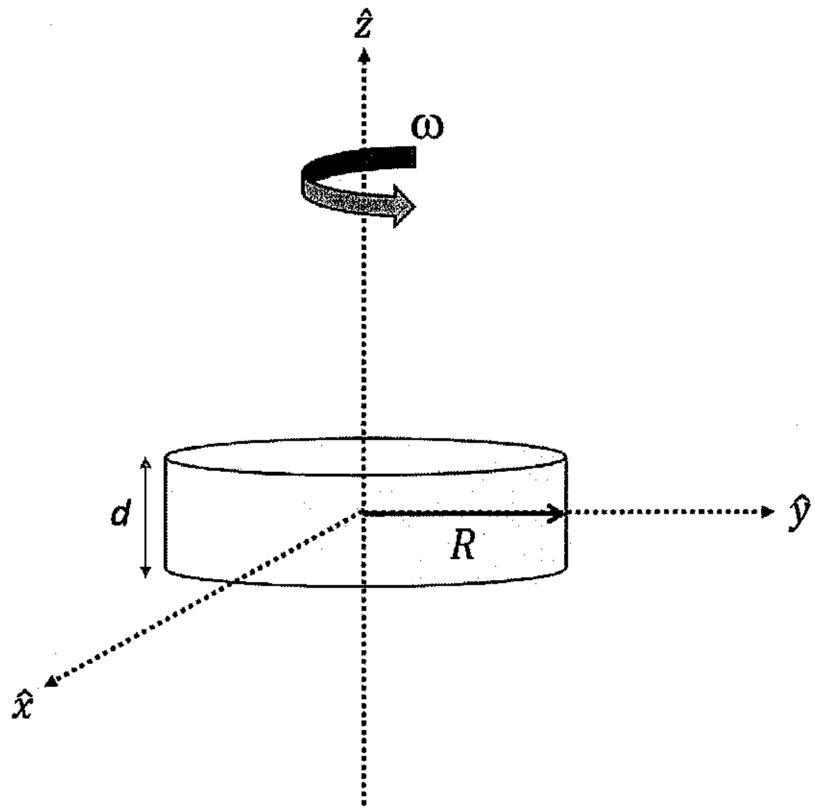


3) A cylindrical wire of length $L$ and radius $R$ carries a uniform current $I$. Use cylindrical coordinates, so the wire occupies the region $0 < z < L$, $r < R$, with $r^2 = x^2 + y^2$. The current flows in the z direction.

a) Neglecting edge effects, calculate the magnetic field at any position inside the cylinder.

b) A beam of charged particles, each of initial momentum $p\hat{z}$ and charge $q$, passes through the cylinder. Neglect any interaction/scattering between the particles and the cylinder material. Assume that the deflection of the charged particles by the magnetic field is small. Show that the beam will focus on a single point after passing through the cylinder. (Since the deflection is small, we have $L \ll$ focal distance.)

4) A rigid disk of radius $R$, thickness $d$, and radially dependent charge density $\rho(r) = \rho_0\left(1 - \frac{r}{R}\right)$, is located with its center at the origin. Here $r^2=x^2+y^2$. The disk is rotating as shown with constant angular velocity $\omega\hat{z}$. Find the magnetic field $\vec{B}(z)$ at points $(0,0,z)$ where $|z| > \frac{d}{2}$



5) Consider a particle of mass, m, in one dimension with potential energy given by

$$V = -V_0 \text{ for } -L < x < L$$

$$V = 0 \text{ otherwise}$$

where $V_0$ is a positive constant.

Consider that the particle has a wavefunction given by:

$$\psi(x) = A(b + x) \text{ for } -b < x < 0$$
$$\psi(x) = A(b - x) \text{ for } 0 < x < b$$
$$\psi(x) = 0 \text{ otherwise}$$

where A and b are positive constants and b > L.

a) First normalize this wavefunction to find the constant, A.

b) Now determine the expectation value of the Hamiltonian. (The kinetic energy is positive; think carefully about the discontinuity in the first derivative of the wavefunction at x=0.)

c) Explore the limit of large b of your results to prove there exists a bound state for this potential.

6) Consider an electron in a magnetic field $B = B_0$ in the z direction. We will consider just the spin degree of freedom. Assume that the electron is in the lowest energy state (spin antiparallel to the external field). At time t=0 an additional magnetic field $B_1$ is turned on along the x direction.

a) Compute the time evolution of the state of the electron in the basis of $S_z$ eigenstates.
b) Compute the mean values of the components $S_z$ and $S_x$ of the electron spin as a function of time.

Hint: $S_x|\uparrow\rangle = \frac{\hbar}{2}|\downarrow\rangle, \quad S_x|\downarrow\rangle = \frac{\hbar}{2}|\uparrow\rangle$

7) Consider an ideal classical diatomic gas whose molecules have an electric dipole moment of magnitude $\mu$. The system consists of N molecules in a box of volume V. The gas is now placed in a uniform applied electric field $E\hat{z}$. Assume that the system is in thermal equilibrium with a heat bath at temperature T. Ignoring interactions between molecules, find the electric polarization $\vec{P}$ .

8) A hot fluid at a temperature $T_1$ flows through the middle of a long, thick pipe of length $L$ with inner radius $R_1$ and outer radius $R_2$. In steady state, the outer surface of the pipe is measured to have a temperature $T_2$. The fluid is in contact with the inner surface of the pipe at $R_1$.

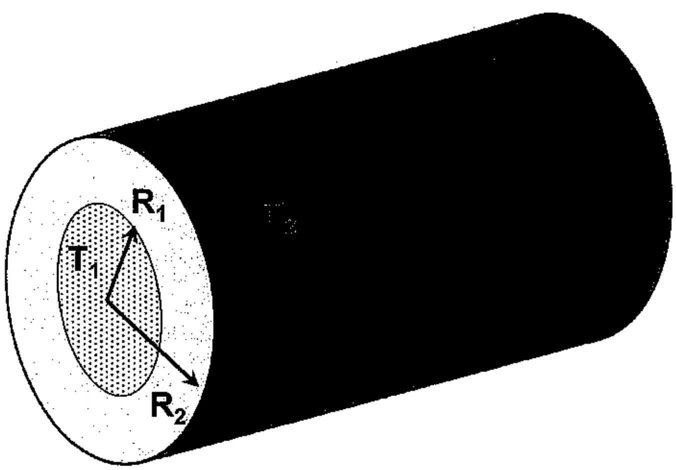
a) Compute T(r) inside the pipe material (i.e. for $R_1 < r < R_2$).

b) At what radius does $T=(T_1+T_2)/2$?

The heat equation in cylindrical coordinates is

$$\rho c \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(k\, r \frac{\partial T}{\partial r}\right)$$

where ρ is the density, c is the specific heat, and k is the thermal conductivity of the pipe material.



# Useful Equations

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad \vec{S}=\frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\int d^3r'\,\frac{J(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$F=-kT\ln Z \qquad Z=\sum_i e^{-\frac{E_i}{kT}} \qquad dF=-SdT-pdV+\mu dN$$

$$i\hbar\frac{\partial\Psi}{\partial t}=H\Psi \qquad -\frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial x^2}+V(x)\psi=E\psi$$

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix} \quad \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\int_0^\infty dx\;x^n e^{-x}=n!$$

# Physics Graduate School Qualifying Examination

## Fall 2017 Part II

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**Instructions:** Work all problems. This is a closed book examination. Calculators may not be used. **Start each problem on a new pack of correspondingly numbered paper and use only one side of each sheet. Place your 3-digit identification number in the upper right hand corner of each and every answer sheet. All sheets, which you receive, should be handed in, even if blank.** Your 3-digit ID number is located on your envelope. All problems carry the same weight.

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1) Two bodies with the same mass *m* are connected with a massless rigid bar of length a. This dumbbell-like system is not subject to any external forces, and its total momentum is zero. The dumbbell initially rotates with angular velocity, $\omega_1$, around its center of mass.

Next, one end collides inelastically with a stationary third body, with the same mass *m*, which sticks to the dumbbell.

a) Derive the angular velocity $\omega_2$ after the collision around the new center of mass of the three bodies.

b) Does the new system rotate faster or slower?

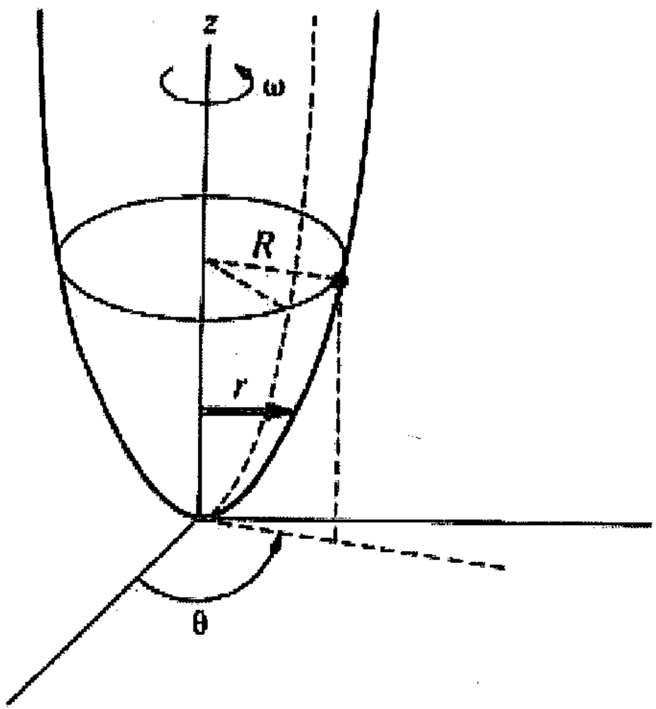
c) Calculate the total kinetic energy of all three bodies, before and after the collision.

2) A bead of mass m slides along a smooth wire bent in the shape of a parabola $z = cr^2 = c(x^2+y^2)$ as shown in the figure below. The wire is rotating about its vertical symmetry axis with angular velocity $\omega$. The gravitational acceleration is g.

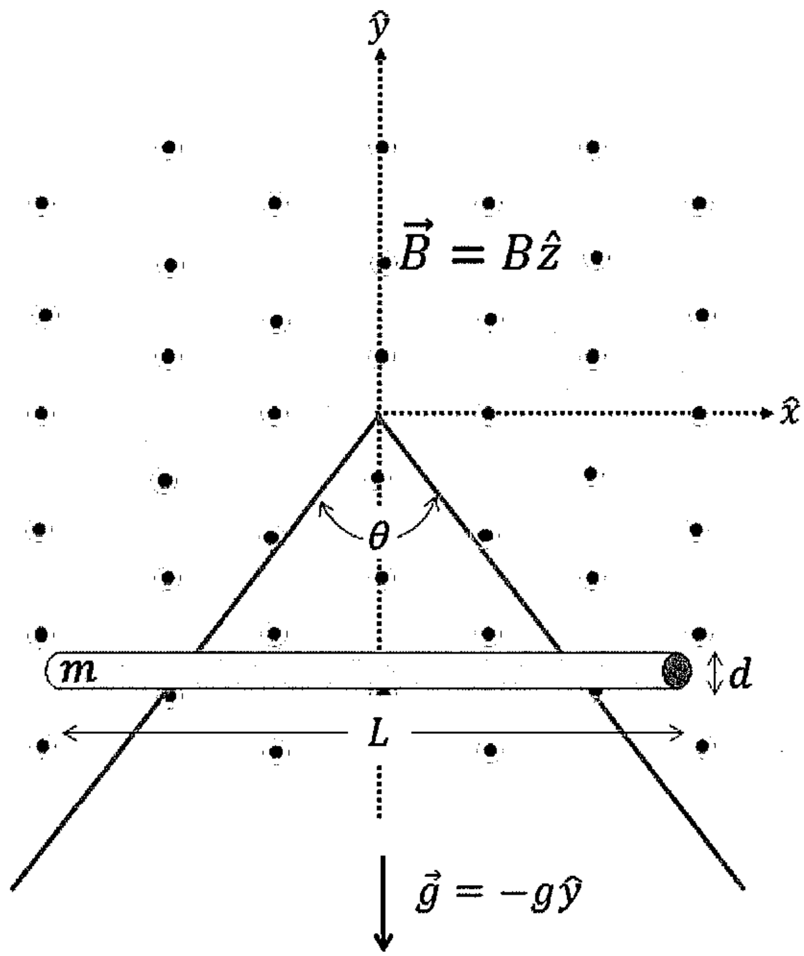
(a) Write down the Lagrangian of the bead in terms of r; the coordinate system is given in the figure.

(b) Express the Lagrange's equation of motion for the bead.

(c) Suppose that the bead's motion is as follows: at constant z and constant r =R, it simply moves in a circle of radius R. Find the value of c.



3) A rigid rod of mass $m$, resistivity $\rho$, total length $L$, diameter $d$ is released from rest from the vertex of a V-shaped rail of negligible electrical resistance. The rail has a vertex angle $\theta$ at the origin as shown in the figure. The system is in the presence of a constant magnetic field $\vec{B} = B\hat{z}$ and acceleration due to gravity is $\vec{g} = -g\hat{y}$. The rod is released at time $t = 0$. Neglecting the effects of friction or air resistance, find an expression for the position of the center of mass of the rod, $\vec{r}(t)$ as a function of time until the rod loses contact with the rails. At what time $t_f$ does the rod lose contact with the rails?



4) (a) An insulating, thin, *hollow* hemisphere (at $x^2+y^2+z^2=R^2$, with z>0) is uniformly charged on its surface; the base is not charged. Prove that the electric field anywhere on the base is perpendicular to the base.
(b) An insulating *solid* hemisphere (filling the region $x^2+y^2+z^2<R^2$, with z>0) is uniformly charged over its volume. Find the component of the electric field on the base that is parallel to the base.

The "base" refers to the set of point z=0, $x^2+y^2<R^2$.

5) Consider a hydrogen atom in a weak external uniform electric field with magnitude, E. The electric field is oriented along the z-axis.

a) Write down the Hamiltonian for this system
b) Consider the electric field as a perturbation to the energy levels of the hydrogen atom. From perturbation theory, the first order correction to the ground state ($n = 1, \ell = 0, m = 0$) energy is

$$\Delta E_{100} = < \psi^0_{100}|H'|\psi^0_{100} >$$

where H' is the perturbation on the Hamiltonian (the electric field term) and the superscript denotes the unperturbed wavefunction. If the ground state of a hydrogen atom is $\frac{1}{\sqrt{\pi}a^{3/2}} e^{-r/a}$ where a is the Bohr radius, then what is the first order correction to the ground state energy?

c) From perturbation theory, the second order correction to the ground state energy is given by

$$\Delta E_{100} = \sum_{nlm \neq 100} \frac{|< \psi^0_{nlm}|H'|\psi^0_{100} > |^2}{E_1 - E_n}$$

Find the correction to the energy, considering only the n=2 terms in the sum.

Use the following. The n=2 hydrogen unperturbed wavefunctions are

$$|\psi_{200} >= \frac{1}{4\sqrt{2\pi}a^{3/2}}\left(2 - \frac{r}{a}\right) e^{-r/(2a)}$$

$$|\psi_{210} >= \frac{1}{4\sqrt{2\pi}a^{3/2}} \frac{r}{a} e^{-r/(2a)} \cos\theta$$

$$|\psi_{21\pm1} >= \frac{1}{8\sqrt{\pi}a^{3/2}} \frac{r}{a} e^{-r/(2a)} \sin\theta \, e^{\pm i\varphi}$$

and the energies are

$$E_n = -\frac{e^2}{8\pi\varepsilon_0 a n^2}$$

6) Consider a quantum mechanical particle in a three dimensional potential such that $V(\lambda r) = \lambda^n V(r)$. (i.e. V(r) = $kr^n$, where k is a constant. Note that $r^2=x^2+y^2+z^2$.) Its Hamiltonian is thus

$$H = \frac{p^2}{2M} + V(r)$$

Define $T_0$ and $V_0$ as the mean values of the kinetic and potential energies in the ground state whose (normalized) wave-function is denoted as $\psi_0(r)$.

If we rescale the wave-function by a factor $\lambda > 0$ to get a new wave-function $\psi_1(r) = A\psi_0(\lambda r)$

a) Find $A$ such that $\psi_1(r)$ is normalized.

b) Compute the mean values of the kinetic ($T_1$), potential ($V_1$) and total energy ($E_1$) in the state $\psi_1(r)$ as a function of $\lambda, T_0, V_0$.

c) For which value of $\lambda$ should the total energy $E_1(\lambda) = \int d^3r\, \psi_1^*(r) H \psi_1(r)$ have a minimum?

d) Use that result to compute a relation between the kinetic and potential energy in the ground state (virial theorem).

7) In this problem we consider a classical ideal gas, made up of N non-interacting entities held at temperature T in a volume V.

a) If the entities making up the gas are atoms, with no internal degrees of freedom, calculate the specific heat at constant volume.

b) Suppose the entities are carbon dioxide molecules. Carbon dioxide, $CO_2$, is a linear molecule with two oxygen atoms located symmetrically on either side of a carbon atom. Treat this molecule as a rigid classical object (which can rotate but does not vibrate) and the atoms as point objects. Calculate the specific heat at constant volume.

Be sure to explain your answers to receive credit.

8) A wire of length *L* and radius *b* is suspended in a vacuum chamber with walls held at a constant temperature $T_o$. A dc electrical current flows through the wire. Assume the predominant energy exchange mechanism between the wire and its environment is thermal radiation.

<u>What is the maximum current that can flow if the wire is not to melt?</u> Assume you know the material properties for the wire given in the table, and that these values do not vary with temperature.

| | |
|---|---|
| Mass density | $\rho_m$ |
| Electrical resistivity | $\rho$ |
| Melting temperature | $T_m$ |
| Specific heat | c |
| Thermal conductivity | k |
| Emissivity | $\varepsilon$ |

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